e-ISSN: 2582-5208
International Research Journal of Modernization in Engineering Technology and Science
(Peer-Reviewed, Open Access, Fully Refereed International Journal)
Volume:03/Issue:10/October-2021 Impact Factor- 6.752 www.irjmets.com

PERT DISTRIBUTION AND ITS PROPERTIES

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ABSTRACT

The PERT distribution is one of the most popular probability continuous distributions with applications to real life data. In this paper some structural properties of this distribution such as Moments, moment generating function, Characteristics function, Cumulative distribution function, Survival function, Hazard function, and also derive the Information matrix. The plots of the Probability density function, cumulative distribution function, Survival function and hazard function of the PERT distribution were constructed for ease of understanding of its shapes under different parameter combinations.

Keywords: PERT Distribution, Moment Generating Function, Survival Function, Hazard Function, Information Matrix.

I. INTRODUCTION

The PERT distribution is widely used in risk analysis to represent the uncertainty of the worth of some quantity where one is counting on subjective estimates, because the three parameters defining the distribution are intuitive to the estimator. In PERT analysis the activity-time distribution is assumed to be a beta distribution, and therefore the mean and variance of the activity time are estimated on the idea of the 'pessimistic', 'most likely' and 'optimistic' completion times, which are subjectively determined by an analyst. In this paper, on the idea of the study of the PERT assumptions, we present an improvement of those estimates. It is also shown that, by means of additional reasonable assumptions, the activity-time distribution in PERT analysis could also be essentially simplified.

PERT helps within the management of the projects by forming a network diagram. The analysis of the network diagram in PERT helps in scheduling the activities associated with the project. Furthermore, using PERT forces the managers involved in the construction industry to organize and quantify project information and allows them to present a graphic display of the project.

In addition, this technique helps in identifying the critical activities involved in the project and in keeping check on those activities that need to be closely monitored. Especially in the case of construction projects, the PERT method would allow the management to have clear views regarding the utilization of processes to maximize the usage of available resources. PERT serves as a key tool to determine the cost, material, time, and requirement of capital related to a specific project.

In probability and statistics, the PERT distribution may be a family of continuous probability distributions defined by the minimum (a), presumably (b) and maximum (c) values that a variable can take. The pdf of PERT Distribution is

Where $\alpha = \dfrac{4b + c - 5a}{c - a} = 1 + \dfrac{4(b - a)}{c - a}$

$$\beta = \frac{5c - a - 4b}{c - a} = 1 + \frac{4(c - b)}{c - a}$$

$$f(x) = \begin{cases} \dfrac{(x-a)^{\alpha-1}(c-x)^{\beta-1}}{\beta(\alpha,\beta)(c-a)^{\alpha+\beta-1}} & if \ \ a \le x \le c \\ 0, \ \ otherwise \end{cases} \qquad (1)$$

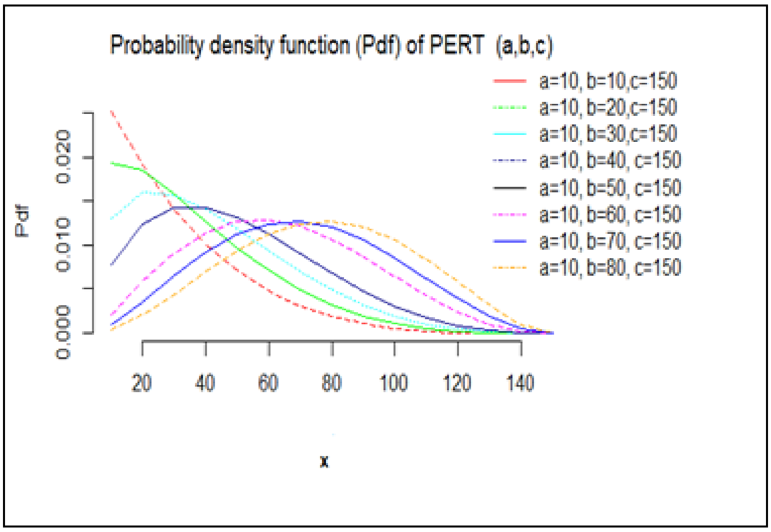

CHART-1.1: Probability Density Function of PERT Distribution (a,b,c)

MOMENTS OF PERT DISTRIBUTION

The Probability density function of PERT Distribution is

$$f(x) = \frac{(x-a)^{\alpha-1}(c-x)^{\beta-1}}{\beta(\alpha,\beta)(c-a)^{\alpha+\beta-1}}, \qquad a \le x \le c$$

The movements of PERT distribution are given by

$$E[X^r] = \int_a^c x^r f(x)dx$$

$$= \frac{1}{\beta(\alpha,\beta)(c-a)^{\alpha+\beta-1}} \int_a^c x^r (x-a)^{\alpha-1}(c-x)^{\beta-1} dx$$

$$E[X^r] = \sum_{k=0}^{r}\left[\binom{r}{k} a^{r-k}.c^k \frac{\Gamma(\alpha+k)\Gamma(\beta+r-k)}{\beta(\alpha,\beta)\Gamma(\alpha+\beta+r)}\right] \quad \text{Where} \quad r \ge k$$

... (2)

Putting r = 1, in equation (2), we get

$$\therefore \quad E(X) = \mu = \left(\frac{a\beta + c\alpha}{\alpha+\beta}\right)^2 = \frac{a+4b+c}{6}$$

Which is the mean μ of the PER distribution **Putting r=2, in the question (2), we get**

$$\therefore E(X^2) = \frac{a^2\beta(\beta+1) + 2ac\alpha\beta + c^2\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$Now Varnance\ \sigma^2 = E(X^2) - E[(X)]^2$$

$$= \frac{a^2\beta(\beta+1) + 2ac\alpha\beta + c^2\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \left(\frac{a\beta+c\alpha}{\alpha+\beta}\right)^2$$

$$\therefore Variance\ \sigma^2 = \frac{(\mu-a)(c-\mu)}{7} \quad or \quad Variance\ \sigma^2 = \frac{(c-a)^2}{36}$$ **Putting r = 3, in the question (2), we get**

$$E(X^3) = \frac{a^3(\beta)_3 + 3a^2c(\alpha)_1(\beta)_2 + 3ac^2(\alpha)_2(\beta)_1 + c^3(\alpha)_3}{(\alpha+\beta)_3}$$ **Putting r = 4, in the question (2), we get**

$$E(X^4) = \frac{a^4(\beta)_4 + 4a^3c(\alpha)_1(\beta)_3 + 6a^2c^2(\alpha)_2(\beta)_2 + 4ac^3(\alpha)_3(\beta)_1}{(\alpha+\beta)_4}$$

MOMENT GENERATING FUNCTION OF PERT DISTRIBUTION

Hyper geometric function: The Poehhammer symbol $(\alpha)_r$ is defined by

$$(\alpha)_r = \alpha(\alpha+1)(\alpha+2)(\alpha+3)..............(\alpha+r-1) \quad (3) \quad = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}, \ where\ r \in Z^+$$

$$and \ (\alpha)_0 = 1$$

The general Hypergeomentric function is

$$_mF_n(\alpha_1, \alpha_2, ...\alpha_m ; \beta_1, \beta_2, ...\beta_n ; x) =$$

$$\sum_{r=0}^{\infty} \frac{(\alpha_1)_r (\alpha_2)_r ......(\alpha_m)_r}{(\beta_1)_r (\beta_2)_r ......(\beta_n)_r} \frac{x^r}{\angle r}, \ where\ m, n \in Z^+$$

If m=1=n, we get

$$_1F_1(\alpha, \beta, x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\beta)_r} \frac{x^r}{\angle r}$$

Which is also denoted by M(α, β, x) and is called confluent Hypergeomentric function

Result:

$$M(\alpha,\gamma,x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 e^{tx}(1-t)^{\gamma-\alpha-1} t^{\alpha-1} dt$$

$$\Rightarrow \int_0^1 e^{tx}(1-t)^{\gamma-\alpha-1} t^{\alpha-1} dt = M(\alpha,\gamma,x)\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} = M(\alpha,\gamma,x)\beta(\alpha,\gamma-\alpha) \quad ... (4)$$

$$Now\ let\ find \ \int_a^c e^{tx} f(x)\, dx$$

$$\int_a^c e^{tx} \frac{(x-a)^{\alpha-1}(c-x)^{\beta-1}}{\beta(\alpha,\beta)(c-a)^{\alpha+\beta-1}} dx$$

$$= \frac{1}{\beta(\alpha,\beta)(c-a)^{\alpha+\beta-1}} \int_a^c e^{tx}(x-a)^{\alpha-1}(c-x)^{\beta-1} dx \therefore \int_a^c e^{tx} f(x) dx = e^{at} \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\alpha+\beta)_r} \frac{(c-a)^r t^r}{\angle r} \quad ...(5)$$

Which is the moment generating function of PERT Distribution

CHARACTERISTIC FUNCTION OF PERT DISTRIBUTION

$$\therefore \int_a^c e^{itx} f(x) dx = e^{ait} \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\alpha+\beta)_r} \frac{(c-a)^r (it)^r}{\angle r} \quad ... (6)$$

CUMULATIVE DISTRIBUTION FUNCTION OF PERT DISTRIBUTION (CDF):

The CDF of PERT Distribution is

$$F(X) = \frac{B_z(\alpha,\beta)}{\beta(\alpha,\beta)} \equiv I_z(\alpha,\beta)$$

$$Where\ z = \left(\frac{x-a}{c-a}\right) and\ B_z\ is\ an\ incomplete\ Beta\ function \therefore\ CDF = \sum_{r=0}^{\infty} \frac{(-1)^r z^{\alpha+r}\,\Gamma(\alpha+\beta)}{\angle r\,(\alpha+r)\Gamma(\beta-r)\,\Gamma(\alpha)} \quad ... (7)$$

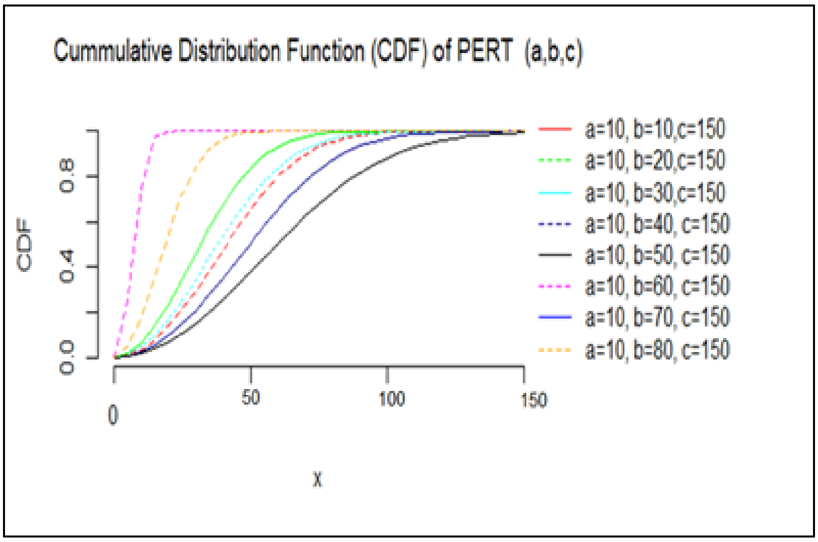

CHART-1.2: Cumulative Distribution Function of PERT Distribution (a,b,c)

SURVIVAL FUNCTION OF THE PERT DISTRIBUTION:

The Survival function = 1-CDF

$$Survival\ function = 1-\sum_{r=0}^{\infty}\frac{(-1)^r\, z^{\alpha+r}\,\Gamma(\beta)}{\angle r\,(\alpha+r)\Gamma(\beta-r)\,\beta(\alpha,\beta)}$$

$$= 1-\frac{\Gamma(\beta)\,z^{\alpha}}{\beta(\alpha,\beta)}\sum_{r=0}^{\infty}\frac{(-1)^r\, z^r}{\angle r\,(\alpha+r)\Gamma(\beta-r)}$$

$$Survival\ function = \frac{\Gamma(\alpha)-\Gamma(\alpha+\beta)\sum_{r=0}^{\infty}\frac{(-1)^r Z^{\alpha+r}}{\angle r(\alpha+r)\Gamma(\beta-r)}}{\Gamma(\alpha)} \quad \ldots (8)$$

$$Where\ \alpha=1+4\left(\frac{b-c}{c-a}\right), \beta=1+4\left(\frac{c-b}{c-a}\right)\ and\ z=\left(\frac{x-a}{c-a}\right)$$

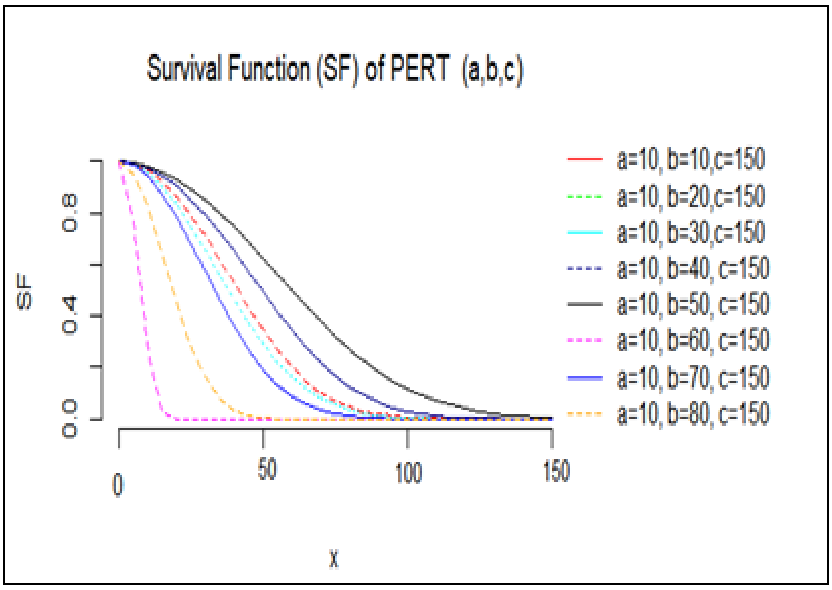

CHART-1.3: Survival Function of PERT Distribution (a,b,c)

HAZARD FUNCTION OF THE PERT DISTRIBUTION:

The Hazard function of the PERT Distribution is

$$Hazard\ \ function = \frac{pdf}{Servival\, function}$$

$$Hazard\ \ function = \frac{\dfrac{(x-a)^{\alpha-1}(c-x)^{\beta-1}}{\beta(\alpha,\beta)(c-a)^{\alpha+\beta-1}}}{1-\dfrac{\Gamma(\beta)z^{\alpha}}{\beta(\alpha,\beta)}\sum_{r=0}^{\infty}\dfrac{(-1)^r z^r}{\angle r(\alpha+r)\Gamma(\beta-r)}} \quad \ldots(9)$$

$$Where\ \alpha = 1+4\left(\frac{b-c}{c-a}\right), \beta = 1+4\left(\frac{c-b}{c-a}\right)\ and\ z = \left(\frac{x-a}{c-a}\right)$$

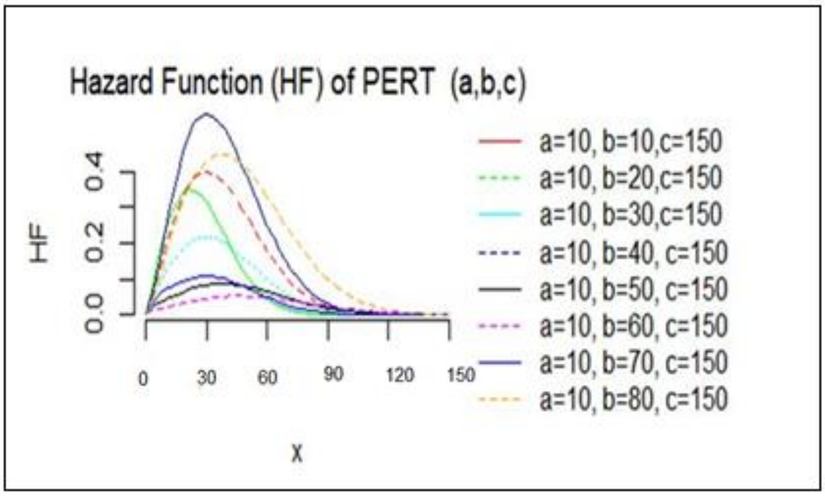

CHART-1.4: Hazard Function of PERT Distribution (a,b,c)

LIKELIHOOD ESTIMATION:

Assume that a random samples x_1, x_2, x_3. ....x_n has been collected from a random variable X which is follows the four parameters of PERT Distribution is

$$f(x,a,c,\alpha,\beta) = \begin{cases} \dfrac{(x-a)^{\alpha-1}(c-x)^{\beta-1}}{\beta(\alpha,\beta)(c-a)^{\alpha+\beta-1}} & if\ \ a \le x \le c \\ 0,\ Otherwise \end{cases} \quad \ldots (10)$$

Where $\alpha > 0$, $\beta > 0$ and $-\infty < a$, $c > 0$

Taking logarithms on both sides form the above likelihood function for a sample of size n for this distribution is

$$\log L(x,a,c,\alpha,\beta) = \log\left[\prod_{i=1}^{n}\left\{\frac{(x_i-a)^{\alpha-1}(c-x_i)^{\beta-1}}{\beta(\alpha,\beta)(c-a)^{\alpha+\beta-1}}\right\}\right]$$

$$= \log\left[\frac{\prod_{i=1}^{n}(x_i-a)^{\alpha-1}(c-xLi)^{\beta-1}}{\{\beta(\alpha,\beta)(c-a)^{\alpha+\beta-1}\}^n}\right]$$

$$= \log L(x,a,c,\alpha,\beta = (\alpha-1)+$$

$$\sum_{i=1}^{n}\log(x_i-a)+(\beta-1)\sum_{i=1}^{n}\log(c-x_i)-n\log[\Gamma(\alpha)]-n\log[\Gamma(\beta)]$$

$$-n\log[\Gamma(\alpha+\beta)]-(\alpha+\beta-1)\log(c-a) \quad \ldots (11)$$

Differentiating (11) partially w.r.t. ‘a’ and equating to ‘0’ as follows

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$$\frac{\partial \log L}{\partial a} = 0$$

$$\Rightarrow (\alpha-1)\sum_{i=1}^{n}\frac{(-1)}{(x_i-a)}+0+n(\alpha+\beta-1)\frac{1}{(c-a)} \Rightarrow (\alpha-1)\sum_{i=1}^{n}\frac{1}{(x_i-a)} = \frac{n(\alpha+\beta-1)}{(c-a)} \qquad \ldots (12)$$

Again differentiating (11) partially w.r.t. 'c' and equating to '0', we get

$$\frac{\partial \log L}{\partial c} = 0 \Rightarrow (\beta-1)\sum_{i=1}^{n}\frac{1}{(c-x_i)} - n(\alpha+\beta-1)\frac{1}{(c-a)} = 0 \; (\beta-1)\sum_{i=1}^{n}\frac{1}{(c-x_i)} = \frac{n(\alpha+\beta-1)}{(c-a)} \qquad \ldots (13)$$

Now Differentiating (11) partially w.r.t 'α' and equating to '0', we get

$$\frac{\partial \log L}{\partial \alpha} = 0$$

$$\Rightarrow \sum_{i=1}^{n}\log(x_i-a) - n\phi(\alpha) - n\phi(\alpha+\beta) - n\log(c-a)$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\log(x_i-a) - \phi(\alpha) - \phi(\alpha+\beta) - \log(c-a) = 0 \qquad \ldots (14)$$

Again Differentiating (13) partially w.r.t. 'β' and equating to '0', we get

$$\frac{\partial \log L}{\partial \beta} = 0$$

$$\Rightarrow \sum_{i=1}^{n}\log(c-x_i) - n\phi(\beta) - n\phi(\alpha+\beta) - n\log(c-a) = 0$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\log(c-x_i) - \phi(\beta) - \phi(\alpha+\beta) - \log(c-a) \qquad (15)$$

Where $\phi(\alpha)$, $\phi(\beta)$, $\phi(\alpha+\beta)$ *are the digamma functions*

INFORMATION MATRIX: The solution of the four equations (12), (13), (14) and (15) gives the maximum likelihood estimator for a, c, α and β. Now we take second partially derivatives of L w.r.t. a, c, α and β, we get.

The solution of the four equations gives maximum Likelihood.

Estimation for a,b,α, and β. The asymptotic variances and covariance of these ML estimates can be obtained from the information matrix, if certain regularity conditions are satisfied, as will be discussed. Taking the various second partial derivatives of L with respect to the parameters gives

$$\frac{\partial^2(\log L)}{\partial a^2}$$

$$= (\alpha-1)\sum_{i=1}^{n}\frac{1}{(x_i-a)} - \frac{n(\alpha+\beta-1)}{(c-a)^2} = \frac{-n\beta(\alpha+\beta-1)}{(\alpha-2)(c-a)^2}$$

$$\frac{\partial^2(\log L)}{\partial c^2}$$

$$= (\beta-1)\sum_{i=1}^{n}\frac{1}{(c-x_i)} - \frac{n(\alpha+\beta-1)}{(c-a)^2} = \frac{-n\alpha(\alpha+\beta-1)}{(\beta-2)(c-a)^2}$$

$$\frac{\partial^2(\log L)}{\partial \alpha^2} = -n\phi(\alpha) - n\phi(\alpha+\beta)$$

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$$\frac{\partial^2(\log L)}{\partial\beta^2} = -n\phi(\beta) - n\phi(\alpha+\beta)$$

$$\frac{\partial^2(\log L)}{\partial a\ \partial c} = \frac{-n(\alpha+\beta-1)}{(c-a)^2}$$

$$\frac{\partial^2(\log L)}{\partial a\ \partial\alpha} = -\sum_{i=1}^{n}\frac{1}{(x-a)} + \frac{n}{(c-a)} = \frac{-n\beta}{(\alpha-1)(c-a)} \qquad \frac{\partial^2(\log L)}{\partial a\ \partial\beta} = \frac{n}{(c-a)}$$

$$\frac{\partial^2(\log L)}{\partial c\ \partial\alpha} = \frac{-n}{(c-a)}$$

$$\frac{\partial^2(\log L)}{\partial c\ \partial\beta} = \frac{n\alpha}{(c-a)(\beta-1)}$$

$$\frac{\partial^2(\log L)}{\partial\alpha\ \partial\beta} = -n\phi'(\alpha+\beta)$$

Where ϕ'(x) is the trigamma function. The information matrix with elements that are negative of expected values of second partial derivative of L is

$$I = Negative\ of\ \ Expected\ values =$$

$$-\begin{bmatrix} \frac{\partial^2(\log L)}{\partial a^2} & \frac{\partial^2(\log L)}{\partial a\ \partial c} & \frac{\partial^2(\log L)}{\partial a\ \partial\alpha} & \frac{\partial^2(\log L)}{\partial a\ \partial\beta} \\ \frac{\partial^2(\log L)}{\partial c\ \partial a} & \frac{\partial^2(\log L)}{\partial c^2} & \frac{\partial^2(\log L)}{\partial c\ \partial\alpha} & \frac{\partial^2(\log L)}{\partial c\ \partial\beta} \\ \frac{\partial^2(\log L)}{\partial\alpha\ \partial a} & \frac{\partial^2(\log L)}{\partial\alpha\ \partial c} & \frac{\partial^2(\log L)}{\partial\alpha^2} & \frac{\partial^2(\log L)}{\partial\alpha\ \partial\beta} \\ \frac{\partial^2(\log L)}{\partial\beta\ \partial a} & \frac{\partial^2(\log L)}{\partial\beta\ \partial c} & \frac{\partial^2(\log L)}{\partial\beta\ \partial\alpha} & \frac{\partial^2(\log L)}{\partial\beta^2} \end{bmatrix}$$

II. CONCLUSION

The PERT distribution produces a bell-shaped curve that is nearly normal. The PERT distribution with unknown end points was investigated as regard maximum likelihood estimation of its parameters. The maximum likelihood equations are derived along with the information matrix. With some assumptions, the information matrix I can be used to establish a minimum variance bound for an unbiased estimator by means of Cramer-Rao inequality. Also, under suitable regularity conditions, consistency and asymptotic normality and efficiency can be claimed for the ML estimates. Then the diagonal elements of I^{-I} are the asymptotic variances of the parameter estimates; however, they turn out to be quite a task to obtain in closed form. Consequently I was inverted numerically for specific values of a, c, b, α, and β.

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