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GENERALIZED FRACTIONAL DIFFERENTIAL OPERATORS INVOLVING MULTIVARIABLE H – FUNCTION

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ABSTRACT

In this paper we use fractional differential operators $D^n_{k,\alpha,x}$ to derive a number of key formulas of multivariable H-function. We use the generalized Leibnitz's rule for fractional derivatives in order to obtain one of the aforementioned formulas, which involve a product of two multi variables H-function. It is further shown that ,each of these formulas yield interesting new formulas for certain multivariable hyper geometric function such as generalized Lauricella function (Srivastava-Dauost) and Lauriella hyper geometric function some of these application of the key formulas provide potentially useful generalization of known result in the theory of fractional calculus.

Keywords: Generalized Fractional Differential Operator, Multivariable H-Function.

I. INTRODUCTION

1.1 Definition

The fractional derivative of special function of one and more variables is important such as in the evaluation of series,[10,15] the derivation of generating function [12,chap.5] and the solution of differential equations [4,14;chap-3] motivated by these and many other avenues of applications, the fractional differential operators $D^n_{k,\alpha,x}$ are much used in the theory of special function of one and more variables.

We use the fractional derivative operator defined in the following manner [5]

$$D^n_{k,\alpha,x}(x^\mu)=\prod_{r=0}^{n-1}\left[\frac{\sqrt{\mu+rk+1}}{\sqrt{\mu+rk-\alpha+1}}\right]x^{\mu+nk} \qquad (1.1)$$

Where≠μ+1 and and k are not necessarily integers

We use the binomial expansion in the following manner

$$(ax^\mu+b)^\lambda=b^\lambda\sum_{l=0}^{\infty}\binom{\lambda}{l}\left(\frac{ax^\mu}{b}\right)^l \qquad where \quad \left[\frac{ax^\mu}{b}\right]<1 \qquad (1.2)$$

Then it is known that the multiple Mellin-Barnes counter integral representing the multivariable H-function converges absolutely under the condition when

$$\xi_i=\min\left\{\mathrm{Re}\left(d_j^{(i)}/\delta_j^{(i)}\right)\right\}, \qquad (j=1,\ldots,m_i)$$
$$\eta_i=\max\left\{\mathrm{Re}\left(c_j^{(i)}-1/\gamma_j^{(i)}\right)\right\}, \qquad (j=1,\ldots,n_i) \qquad (1.3)$$

II. GENERALIZED FRACTIONAL DIFFERENTIATION OPERATORS

The fractional calculus operator involving various special functions, have been found significant importance and applications in various sub-field of application mathematical analysis. Since last five decades, a number of workers like Srivastava et al. [21], Saigo [10] etc. have studied in depth, the properties, applications and different extensions of various hyper geometric operators of fractional calculus.

Let $\alpha,\alpha',\beta',\gamma\epsilon C,\ \gamma=0\ \ and\ x\epsilon R_+,$ then the generalized fractional differentiation operators [25] involving Appell function F_3 as a kernel are defined by the following equations:

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$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{0+}^{-\alpha',-\alpha,\beta',\beta,-\gamma} f\right)(x) \tag{2.1}$$

$$= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{-\alpha',-\alpha,\beta',n,-\gamma+n} f\right)(x), \qquad (Re\ (\gamma) > 0; n = [Re\ (\gamma)] + 1\ ) \tag{2.2}$$

$$= \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dx}\right)^n \left(x^{\alpha'}\right) \int_0^x (x-t)^{n-\gamma-1}\ t^{\alpha}\ F_3\left(-\alpha',-\alpha,n-\beta',-\beta,n,-\gamma;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)dt, \tag{2.3}$$

And

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{-}^{-\alpha',-\alpha,\beta',\beta,-\gamma} f\right)(x) \tag{2.4}$$

$$= \left(-\frac{d}{dx}\right)^n \left(I_{-}^{-\alpha',-\alpha,\beta',n,-\gamma+n} f\right)(x),\ (Re\ (\gamma) > 0; n = [Re\ (\gamma)] + 1\ ) \tag{2.5}$$

$$= \frac{1}{\Gamma(n-\gamma)} \left(-\frac{d}{dx}\right)^n \left(x^{\alpha}\right) \int_0^{\infty} (t-x)^{n-\gamma-1}\ t^{\alpha'}\ F_3\left(-\alpha',-\alpha,n-\beta',-\beta,n,-\gamma;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)dt, \tag{2.6}$$

where $I_{0+}^{\alpha',\alpha,\beta',\beta,\gamma}$ *and* $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ are Saigo-Maeda fractional integral operators, and Appell hyper geometric function of two variables is defined as

$$F_3(\alpha,\alpha',\beta,\beta';\gamma;z,\xi) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{z^m}{m!}\frac{\xi^n}{n!}, \quad (|z| < ,1|\xi| < 1); \tag{2.7}$$

where $(z)_m$ *and* $(z)_n$ are the Pochhammer symbol defined by

$z \epsilon C$ *and* $m,n \epsilon\ N_0 = N \cup \{0\}, N = (1,2,3,\ldots)$ *by* $(z)_0 = 1, (z)_m = z(z+1)\ldots(z+m-1).$

The series in (2.7) is absolutely convergent for

$$(|z| < ,1|\xi| < 1)\ \text{and}\ (|z| = ,1|\xi| = 1), \qquad \text{where}\ (z,\xi \neq 1).$$

These operators reduce to Saigo derivative operators [25] as

$$\left(D_{0+}^{0,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(D_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f\right)(x), \quad (Re(\gamma) > 0); \tag{2.8}$$

$$\left(D_{-}^{0,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(D_{-}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f\right)(x), \quad (Re(\gamma) > 0); \tag{2.9}$$

Further [25, p. 394, Eqns. (4.18) and (4.19)] we also have

$$I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} = \Gamma\left[\begin{matrix}\rho,\rho+\gamma-\alpha-\alpha'-\beta,\rho+\beta'-\alpha' \\ \rho+\gamma-\alpha,-\alpha',\rho+\gamma-\alpha'-\beta,\rho+\beta'\end{matrix}\right] x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{2.10}$$

Where $Re(\gamma) > 0,\ Re(\rho) > \max[0, Re(\alpha+\alpha'+\beta-\gamma), Re(\alpha'-\beta')],$ *and*

$$I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1} = \Gamma\left[\begin{matrix}1+\alpha+\alpha'-\gamma-\rho,1+\alpha+\beta'-\gamma-\rho,1-\beta-\rho \\ 1-\rho,1+\alpha+\alpha'+\beta'-\gamma-\rho,1+\alpha-\beta-\rho\end{matrix}\right] x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{2.11}$$

Where $Re(\gamma) > 0,\ Re(\rho) > \min[Re-(-\beta), Re(\alpha,\alpha',-\gamma), Re(\alpha+\beta'-\gamma)],$

Here, we have used the symbol $\Gamma\left[\begin{matrix}\cdots\\ \cdots\end{matrix}\right]$ representing the fraction of many Gamma functions.

III. MAIN RESULT

Throughout the present paper .we assume that the convergence and existence condition corresponding appropriately to the ones detained above are satisfied by each of the various H-function involved in our results which are presented in the following sections

Generalized Fractional differential operator involving multivariable H-function

In this section we shall prove our main formulas on fractional differential operator involving multivariable H-function

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Theorem-1

$\Rightarrow D^{n}_{k,\alpha,x}\{x^{\lambda}(x^{v_1}+a)^{\lambda}(b-x^{v_2})^{-\delta} H[z_1 x^{\rho_1}(x^{v_1}+a)^{\sigma_1}(b-x^{v_2})^{-\delta_1} \ldots \ldots$

$z_r x^{\rho_r}(x^{v_1}+a)^{\sigma_r}(b-x^{v_2})^{-\delta_r}]\}$

Provided (in addition to the appropriate convergence and existence condition) that

$$\min\{\upsilon_1, \upsilon_2, \rho_i, \sigma_i, \delta_i\} > 0 \;(i=1,\ldots,r)$$

$$\max\left\{\left|\arg\left(x^{\upsilon_1}/a\right)\right|, \left|\arg\left(x^{\upsilon_2}/b\right)\right|\right\} < \pi$$

$$\operatorname{Re}(k) + \sum_{i=1}^{r} \rho_i \xi_i > -1$$

Whee $\xi_i = (i=1,\ldots,r)$ are given in (1.3)

$$\Rightarrow a^{\lambda} b^{-\delta} x^{k+nk} \sum_{l,m=0}^{\infty} \frac{\left(\frac{x^{v_1}}{a}\right)^l \left(\frac{x^{v_2}}{b}\right)^m}{\lfloor l \lfloor m} H^{0,\; n'+n+2;\; m_1 n_1 \ldots m_r, n_r}_{p'+n+2,\; q'+n+2;\; p_1 q_1 \ldots p_r q_r}$$

$$\times \left[\begin{matrix} z_1 x^{\rho_1} a^{\sigma_1} b^{-\delta_1} \\ \\ z_r x^{\rho_r} a^{\sigma_r} b^{-\delta_r} \end{matrix}\right| \begin{matrix} (-\lambda, \sigma_1, \ldots \sigma_r)(-\delta-m; \delta_1 \delta_2 \ldots \delta_r), (-k-kr-v_1 v_2 m;\, \rho_1 \ldots \rho_r) 0, n-1 \\ \\ (-\lambda+1, \sigma_1 \sigma_2, \ldots \sigma_r)(1-\delta, \delta_1 \ldots \delta_r), (\alpha-k-rk-v_1 l v_2 m;\, \rho_1 \ldots \rho_r) 0, n-1 \end{matrix}$$

$$\left.\begin{matrix} (a_j,\; \alpha_j^1 \ldots \alpha_j^{(r)})_{1,p} : (c_j^1 r_j^1)_{1,p_1} \ldots (c_j^{(r)}, r_j^{(r)})_{1,p_r} \\ \\ (b_j,\; \beta_j^1 \ldots \beta_j^{(r)})_{1,q} : (d_j^1\, \delta_j^1)_{1,q_1} \ldots (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \end{matrix}\right] \qquad (3.1)$$

Theorem-2

$$D^{n}_{k,\alpha,x}\{x^{k}(x^{v_1}+a)^{\lambda}(b-x^{v_2})^{\delta} H[z_1 x^{\rho_1}(x^{v_1}+a)^{\sigma_1}(b-x^{v_2})^{-\delta_1} \ldots \ldots \ldots z_r x^{\rho_r}$$

$$(x^{v_1}+a)^{\sigma_r}(b-x^{v_2})^{-\delta_r}] H^{*}[w_1 x^{\lambda_1} \ldots \ldots w_s x^{\lambda_s}]\}$$

Provided (in addition to the appropriate convergence and existence conditions mentioned with (3.1)

$$\Rightarrow a^{\lambda} b^{-\delta} x^{k+nk} \sum_{l,m=0}^{\infty} \frac{\left(\frac{x^{v_1}}{a}\right)^l \left(\frac{x^{v_2}}{b}\right)^m}{\lfloor l \lfloor m} H^{0,\; n'+n+3+N,\; m_1, n_1 : \ldots m_r, n_r, M_1, N_1 \ldots M_s, N_s}_{p'+P+n+3, q'+Q+n+3,\; p_1, q_1, \ldots p_r, q_r\, P_1, Q_1 \ldots P_s, Q_s}$$

$$\left[\begin{matrix} z_1 a^{\sigma_1} b^{-\delta_1} x^{\rho_1} \\ \\ z_r a^{\sigma_r} b^{-\delta_r} x^{\rho_r} \end{matrix}\right| \begin{matrix} (-\lambda_1 \sigma_1 \ldots \ldots \sigma_r)\,(1-\delta-m; \delta_1 \ldots \ldots \delta_r)\,(a_j\,;\, \alpha_j^1 \ldots \ldots \alpha_j^r)_{1,p} (c_j^i\;\, r_j^i)_{1,p_i} \\ \\ (-\lambda_1+l, \sigma_1 \ldots \ldots \sigma_r)(1-\delta;\, \delta_1 \ldots \ldots \delta_r)(b_j\,;\, \beta_j^1 \ldots \ldots \beta_j^r)_{1,q} (d_j^i\;\, \delta_j^i)_{1,q_i} \end{matrix}$$

$$\left.\begin{matrix} \ldots\ldots(c_j^r\ r_j^r)_{1,p_r}\left(-k-r'k-v_1l-v_2m,\rho_1\ \rho_2\ldots\ldots\rho_r,\lambda_1\ldots\ldots\lambda_s\right)_{0,n-1} \\ \\ \ldots\ldots(d_j^r\ \delta_j^r)_{1,q_r}\left(\alpha-k-r'k-v_1l-v_2m,\rho,\rho_2\ldots\ldots\rho_r,\lambda_1\ldots\ldots\lambda_s\right)_{0,n-1} \end{matrix}\right] \quad (3.2)$$

Theorem-3

$$D_{k,\alpha,x}^{n}\left\{x^{k}(x^{v_1}+a)^{\lambda}\,(b-x^{v_2})^{-\delta}\,H[z_1x^{\rho_1}\ldots\ldots z_rx^{\rho_r}]\right\}$$

Provided (in addition to the appropriate convergence and existence conditions mentioned with (3.1)

$$\Rightarrow a^{\lambda}b^{-\delta}x^{k+nk}\sum_{l,m=0}^{\infty}\left(\frac{x^{v_1}}{a}\right)^{l}\left(\frac{x^{v_2}}{b}\right)^{m}\binom{\lambda}{l}\binom{\delta}{m}H^{0,n+n':\ m_1,n_1,\ldots\ldots,m_r,n_r}_{p+n';q+n'\ p_1,q_1,\ldots\ldots,p_r,q_r}$$

$$\left[\begin{matrix} z_1x^{\rho_1} \\ \vdots \\ z_rx^{\rho_r} \end{matrix}\left|\begin{matrix} (-k-kv-v_1l-v_2m;\rho_1\ldots\ldots\rho_r)_{t=0,n-1}\ (a_j;\alpha_j^{(1)};\ldots\ldots\alpha_j^{(r)})_{1,p}(c_j^{1},r_j^{1})_{1,p_1}\ldots\ldots(c_j^{(r)}\ldots\ldots v_j^{(r)})_{1,p_r} \\ \\ (\alpha-tk-k-v_1l-v_2m;\rho_1\ldots\ldots\rho_r)_{t=0,n-1}\ (b_j;\beta_j^{1}\ldots\ldots\beta_j^{(r)})_{1,q}(d_j^{1},\delta_j^{1})_{1,q_1}\ldots\ldots(d_j^{(r)}\ldots\ldots\delta_j^{(r)})_{1,q_r} \end{matrix}\right.\right] \quad (3.3)$$

Proof:- We first replace the multivariable H-function occurring on the LHS by its Mellin Barnes contour integrals collected the powers of x, $\left(x^{\upsilon_1}+a\right)$ and $\left(b-x^{\upsilon_2}\right)$ and apply binomial expansion

$$(x+\xi)^{\lambda}=\xi^{\lambda}\left(1+\frac{x}{\xi}\right)^{\lambda}=\xi^{\lambda}\sum_{l=0}^{\infty}\binom{\lambda}{l}\left(\frac{x}{\xi}\right)^{l};\qquad\left|\frac{x}{\xi}\right|<1 \quad (3.4)$$

We then apply the fractional derivative operator in the following manner [5]

$$D_{k,\alpha,x}^{n}\left(x^{\mu}\right)=\prod_{r=0}^{n-1}\left[\frac{\Gamma\mu+rk+1}{\Gamma\mu+rk-\alpha+1}\right]x^{\mu+nk} \quad (3.5)$$

Where $\alpha\neq\mu+1$ and α and k are not necessarily integers and interpret the resulting MillenBarnes contour integrals as a H-function of r-variables we shall arrive at(3.1)

Proof of (3.2):-

We first replace the multivariable H-function occurring on the LHS by its Mellin Barnes contour integrals collected the powers of x, $\left(x^{\upsilon_1}+a\right)$ and $\left(b-x^{\upsilon_2}\right)$ and apply binomial expansion

$$(x+\xi)^{\lambda}=\xi^{\lambda}\left(1+\frac{x}{\xi}\right)^{\lambda}=\xi^{\lambda}\sum_{l=0}^{\infty}\binom{\lambda}{l}\left(\frac{x}{\xi}\right)^{l};\qquad\left|\frac{x}{\xi}\right|<1 \quad (3.4)$$

We then apply the fractional derivative operator in the following manner [5]

$$D_{k,\alpha,x}^{n}\left(x^{\mu}\right)=\prod_{r=0}^{n-1}\left[\frac{\Gamma\mu+rk+1}{\Gamma\mu+rk-\alpha+1}\right]x^{\mu+nk} \quad (3.5)$$

Where $\alpha\neq\mu+1$ and α and k are not necessarily integers and interpret the resulting MillenBarnes contour integrals as a H-function of r & s variables we shall arrive at (3.2)

Proof of (3.3):- same as proof of theorem (3.1)

IV. CONCLUSION

In this paper we use fractional differential operators $D_{k,\alpha,x}^{n}$ to derive a number of key formulas of multivariable H-function. We use the generalized Leibnitz's rule for fractional derivatives in order to obtain one of the aforementioned formulas, which involve a product of three multivariables H-function. It is further shown that ,each of these formulas yield interesting new formulas for certain multivariable hypergeometric function such as generalized Lauricella function (Srivastava-Dauost)and Lauriella hyper geometric function some of these application of the key formulas provide potentially useful generalization of known result in the theory of fractional calculus.

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