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ENHANCING THE NUMERICAL SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS THROUGH HYBRID BERNSTEIN POLYNOMIAL METHODS

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ABSTRACT

Partial Differential Equations (PDEs) are fundamental tools in the mathematical modeling of phenomena across various scientific and engineering disciplines, including fluid dynamics, electromagnetic theory, and financial mathematics. Traditionally, solutions to these equations are sought through analytical methods or numerical techniques such as finite differences, finite elements, and spectral methods. However, these conventional approaches often struggle with complex geometries, nonlinearities, and the need for high precision across broad domains. Bernstein polynomials have emerged as a powerful alternative for numerically solving PDEs. Their properties—non-negativity, partition of unity, and ease of differentiation and integration—make them particularly suitable for approximation tasks. Bernstein polynomials are foundational in computer-aided geometric design due to their strong stability and best-approximation characteristics under certain norms. Hybrid methods that integrate Bernstein polynomials with numerical techniques like collocation, Galerkin, or tau methods have demonstrated enhanced convergence rates and accuracy. These methods have been successfully applied to a wide range of problems, from simple heat transfer to complex fluid flows, achieving reduced computational costs and improved precision. Despite some challenges, such as determining the optimal polynomial degree and managing computational efficiency, ongoing research continues to address these issues, broadening the applicability and effectiveness of Bernstein polynomials in solving PDEs.

Keywords: Bernstein Polynomial, Finite Difference, Partial Differential Equation.

I. INTRODUCTION

Partial Differential Equations (PDEs) form the cornerstone of mathematical modeling of phenomena in various scientific and engineering disciplines, encompassing areas such as fluid dynamics, electromagnetic theory, and financial mathematics. The solutions to these equations often provide insights into the dynamics of systems and are crucial in predicting behaviors in real-world scenarios.

Classically, PDEs are tackled using analytical methods when possible, and otherwise, numerical methods such as finite differences, finite elements, and spectral methods are employed. However, there are limitations to these traditional approaches, particularly when dealing with complex geometries, nonlinearities, and problems requiring high accuracy over global domains.

In recent years, the use of Bernstein polynomials has gained prominence as a powerful tool in the numerical solution of PDEs. Bernstein polynomials possess several advantageous properties, such as non-negativity, partition of unity, and ease of differentiation and integration, which make them suitable for approximating solutions to PDEs.

The literature indicates that Bernstein polynomials are particularly effective due to their geometric representation and approximation capabilities. These polynomials form the basis of Bezier curves and surfaces, which are paramount in computer-aided geometric design (CAGD). Their strong stability properties and the ability to form a 'best approximation' under certain norms have been well-documented.

Researchers have integrated Bernstein polynomials with numerical methods to develop hybrid approaches. These methods often involve representing the solution as a Bernstein polynomial series and then discretizing the PDE using techniques such as collocation, Galerkin, or tau methods. This approach has been found to enhance the convergence rates and provide highly accurate solutions.

Applications in various fields have been explored, ranging from simple heat transfer problems to complex fluid flows characterized by shocks and discontinuities. Studies have shown that the use of Bernstein polynomials can lead to reduced computational costs and improved accuracy.

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However, there are challenges associated with this approach. The degree of the polynomial can significantly affect the computational efficiency, and the choice of the degree is not always straightforward. Moreover, while Bernstein polynomials ensure local control of the approximation, their global support can lead to a dense system matrix, which may pose computational challenges.

Recent advancements have addressed some of these challenges. Adaptive methods that adjust the polynomial degree based on the solution's behavior and domain decomposition techniques that localize the problem have been proposed. Furthermore, the development of sparse Bernstein polynomial techniques has been a significant step toward mitigating the computational intensity. In conclusion, Bernstein polynomials represent a promising direction for the numerical solution of PDEs, offering advantages in stability, accuracy, and geometric interpretation. While challenges remain, ongoing research continues to refine these methods, expanding their appli- cability and efficiency. The combination of Bernstein polynomials with traditional numerical techniques holds the potential to address some of the most challenging problems posed by PDEs in modern computational science.

II. BERNSTEIN POLYNOMIALS AND NUMERICAL METHOD

Bernstein polynomials

Bernstein polynomials represent a specific group of polynomials with significant roles in approximation theory and the field of computer graphics. These polynomials are notably utilized in formulating B´ezier curves.

A Bernstein polynomial of degree n is defined for k = 0, 1, 2, , n as :

$$B_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k}$$

where

- $B_{k,n}(t)$ is k^{th} bernstein polynomial of degree n.
- $\binom{n}{k}$ is a binomial coefficient, calculated as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
- t is variable, within the interval $[0, 1]$.

Some properties of bernstein polynomials:

1. $B_{k,n}(t)$ is non-negative within the interval [0, 1].
2. $B_{k,n}(t)$ is continuous function on [0, 1].
3. The sum of Bernstein polynomials of degree n is 1 for every $t \in [0, 1]$.
4. Bernstein polynomials are symmetric in the sense that

$$B_{k,n}(t) = B_{n-k,n}(1 - t).$$

5. $B_{k,n}(t)$ can be written in the form of recursive relation:

$B_{k,n}(t) = (1 - t)B_{k,n-1}(t) + tB_{k-1,n-1}(t)$.

2.1 Proposed method:

The methodology for solving partial differential equations (PDEs) using a hybrid method that combines Bernstein polynomials with traditional numerical techniques, such as finite difference or finite element methods, can be quite intricate due to the complex nature of PDEs. This approach is designed to leverage the strengths of both polynomial approximation (for smooth, global representation) and numerical discretization methods (for handling complex geometries and boundary conditions). Let's discuss this methodology in detail:

2.2 Problem statement

Consider a partial differential differential equation of the form

$$L[u(x, y)] = f(x, y),$$

where L is a differential operator, u(x, y) is unknown function, and f (x, y) is a given function.

2.3 Initial approximation with Bernstein Polynomials

The next step is to approximate the solution u(x, y) using a series of Bernstein polynomials. Bernstein polynomials are particularly useful for their properties of stability and good approximation capabilities. The approximation is typically expressed as:

$$u(x,y) \approx \sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij} B_{i,n}(x) B_{j,n}(y)$$

where Bk,n(z) is the Bernstein polynomial of degree n and aij are the coefficients to be determined.

1. **Discretization using Numerical Methods**

The domain of the PDE is discretized into a grid or mesh, and a numerical method (like finite difference) is applied to approximate the derivatives in the PDE at discrete points. This step transforms the continuous PDE into a discrete form that can be solved computationally.

2. **Combining Both Approaches**

The Bernstein polynomial approximation is substituted into the discretized form of the PDE at each grid point or within each mesh element. This integration of the polynomial approximation with the numerical method leads to a system of equations that govern the Bernstein coefficients aij.

3. **Solving the System for Bernstein Coefficients**

The resulting system of equations (which could be linear or nonlinear) is solved using numerical methods such as iterative solvers, Newton-Raphson method, or Gaussian elimination. This step determines the values of the Bernstein coefficients aij that best fit the PDE under the chosen discretization.

4. **Iterative Refinement and Error Analysis**

The accuracy of the solution is evaluated, and error analysis is performed. If necessary, adjustments are made to the degree of the Bernstein polynomial or the discretization parameters. An iterative process is applied, where the solution is refined by progressively adjusting the polynomial degree or the discretization scheme until the desired accuracy is achieved.

5. **Validation and Application**

Finally, the obtained solution is validated against known solutions or through numerical simulations. The method is tested on various types of PDEs to assess its versatility and limitations.

Benefits and Challenges: The hybrid method combines the global approximation capabilities of Bernstein polynomials with the local precision of numerical discretization methods, offering a balance between accuracy and computational efficiency. However, the approach can be computationally intensive, especially for higher degrees of polynomials and finer discretizations, and may require sophisticated numerical techniques to handle nonlinearity and stability issues.

III. NUMERICAL RESULTS

This section includes the implementation and analysis of proposed method to solve PDE. The obtained solution is compared with exact solution.

Example 3.1. *Consider the Advection equation with BVP:*

$$2v_t + 3v_x = 0, \quad 0 < x < 3,$$

$$v_0(x) = \sin(5\pi x), \quad 0 < x < 3, \quad v(0,t) = \sin(3t).$$

Firstly, write the PDE in the standard form $u_t + \frac{3}{2}u_x = 0$. *The exact solution to the BVP is given as,*

$$v(x,t) = \begin{cases} \sin\left(5\pi\left(x - \frac{3t}{2}\right)\right) & 0 < x < 3, \quad t > 0, \\ \sin\left(3\left(t - \frac{2x}{3}\right)\right) & 0 < x < \min\left(\frac{3t}{2}, 3\right) \quad and \quad t > 0. \end{cases}$$

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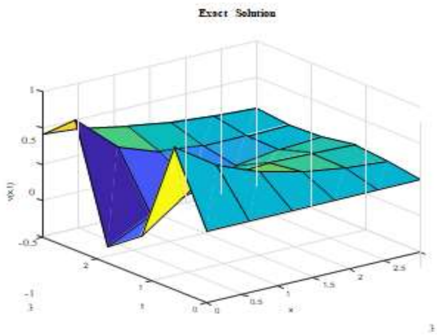
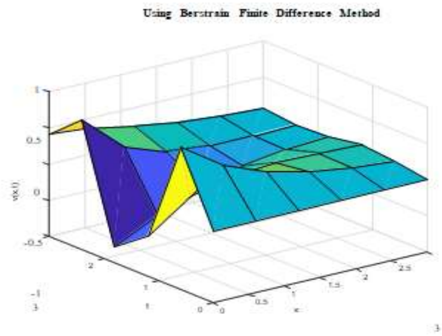
The PDE is discretized and solved using Bernstein finite difference method. This method involves creating a system of equations based on the PDE and solving it for the coefficients of the Bernstein polynomials, which approximate the solution. A surface plot visualizes the solution obtained by this method. The error obtained is 3.8048e−10.

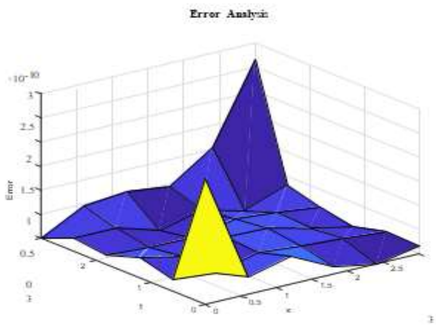
Exact Solution Using Berstrain Finite Difference Method

Error Analysis





1a: 1b:



1c:

Fig 1: Advection equation:

IV. CONCLUSION

In summary, Bernstein polynomials present a promising approach for the numerical solution of PDEs, offering benefits in terms of stability, accuracy, and geometric interpretation. While the computational intensity and polynomial degree selection pose challenges, recent advancements in adaptive methods and domain decomposition techniques have addressed some of these issues. The development of sparse Bernstein polynomial techniques has further mitigated computational demands. The hybrid method, which combines Bernstein polynomials with traditional numerical techniques, balances global approximation capabilities with local precision, providing a robust solution for complex PDEs. Ongoing research continues to refine these methods, expanding their applicability and efficiency in modern computational science. By integrating Bernstein polynomials with established numerical techniques, this approach holds potential for addressing some of the most challenging problems in PDEs.

V. REFERENCES

[1] K.B. OldhamandJ. Spanier, Theory and Applications of Differentiation and Integration to Arbitrary Order. New York, NY, USA: Academic, 1974.

[2] F. Jarad and T. Abdel jawad, “A modified Laplace transform for certain generalized fractional operators,” Results Nonlinear Anal., vol. 1, no. 2, pp. 88–98, 2018.

[3] J. H. He, “Homotopy perturbation method for solving boundary value problems,” Phys. Lett. A, vol. 350, pp. 87–88, Jan. 2006.

[4] S. Yang, A. Xiao, and H. Su, “Convergence of the variational iteration method for solving multi-order

fractional differential equations," Comput. Math. Appl., vol. 60, no. 10, pp. 2871–2879, Nov. 2010.

[5] R.Shah, H.Khan, U.Farooq, D.Baleanu, P.Kumam, and M.Arif, "Anew analytical technique to solve system of fractional-order partial differential equations," IEEE Access, vol. 7, pp. 150037–150050, 2019.

[6] J.-S.Duan, T.Chaolu, R.Rach, and L.Lu, "The adomian decomposition method with convergence acceleration techniques for nonlinear fractional differential equations," Comput. Math. Appl., vol. 66, no. 5, pp. 728–736, Sep. 2013.

[7] R. Jafari, W. Yu, and X. Li, "Fuzzy differential equations for nonlinear system modeling with Bernstein neural networks," IEEE Access, vol. 4, pp. 9428–9436, 2016.

[8] S. Noinang, M. Munawar, M. A. Z. Raja, Z. Sabir, T. Botmart, W. Weera, and P. Junsawang, "Numerical assessments employing neural networks for a novel drafted anti-virus subcategory in a nonlinear fractional-order SIR differential system," IEEE Access, vol. 10, pp. 114192–114202, 2022.

[9] H. Tajadodi, H. Jafari, and M. N. Ncube, "Genocchi polynomials as a tool for solving a class of fractional optimal control problems," Int. J. Optim. Control, Theories Appl., vol. 12, no. 2, pp. 160–168, Jul. 2022.

[10] Y. Xie, L. Li, and M. Wang, "Adomian decomposition method with orthogonal polynomials: Laguerre polynomials and the second kind of Chebyshev polynomials," Mathematics, vol. 9, no. 15, p. 1796, Jul. 2021, doi: 10.3390/MATH9151796.

[11] M. Yavuz and N. O¨ zdemir, "A different approach to the European option pricing model with new fractional operator," Math. Model. Natural Phenomena, vol. 13, no. 1, p. 12, 2018.

[12] M. Yavuz and N. Ozdemir, "Numerical inverse Laplace homotopy technique for fractional heat equations," Thermal Sci., vol. 22, no. 1, pp. 185–194, 2018.

[13] Y.-N. Zhang, Z.-Z. Sun, and H.-L. Liao, "Finite difference methods for the time fractional diffusion equation on non-uniform meshes," J. Comput. Phys., vol. 265, pp. 195–210, May 2014.

[14] S.N.Tural-PolatandA.T.Dincel,"Numericalsolutionmethodformulti- term variable order fractional differential equations by shifted Chebyshev polynomials of the third kind," Alexan- dria Eng. J., vol. 61, no. 7, pp. 5145–5153, Jul. 2022.

[15] M. H. T. Alshbool, M. Mohammad, O. Isik, and I. Hashim, "Fractional Bernstein opera- tional matrices for solving integro-differential equations involved by caputo fractional deriva- tive," Results Appl. Math., vol. 14, May 2022, Art. no. 100258.

[16] V. Cichella, I. Kaminer, C. Walton, N. Hovakimyan, and A. M. Pascoal, "Consistent ap- proximation of optimal control problems using Bernstein polynomials," in Proc. IEEE 58th Conf. Decis. Control (CDC), Dec. 2019, pp. 4292–4297.

[17] K. Maleknejad, E. Hashemizadeh, and R. Ezzati, "A new approach to the numerical solution of Volterra integral equations by using Bernstein's approximation," Commun. Nonlinear Sci. Numer. Simul., vol. 16, no. 2, pp. 647–655, Feb. 2011.