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# POLYNOMIAL EQUATIONS OF BICOMPLEX NUMBERS

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## ABSTRACT

In this paper, a polynomial of a bicomplex variable has been studied. Square roots for several quadratic equations using bicomplex variables have been obtained. Also, we have studied the nth root of unity in $\mathbb{C}_2$. We have shown that all $n^2$ nth roots of unity in $\mathbb{C}_2$ form an abelian group. The number of distinct cyclic subgroups within an abelian group of the nth root of unity in $\mathbb{C}_2$ has also been characterized.

**Keywords:** Bicomplex Numbers, Square Roots, Nth Root Of Unity.

**AMS Subject Classification:** 30G35, 32A30, 30C15, 12D10, 32A08.

## I. INTRODUCTION

Throughout this paper, the sets of real and complex numbers are denoted by $\mathbb{C}_0$ and $\mathbb{C}_1$, respectively, while the set of bicomplex numbers is denoted by $\mathbb{C}_2$. The theory of bicomplex numbers have been introduced in detail by **Price (1991)** and **Luna-Elizarrarás et al., (2015)**. Some glimpses of the theory can be seen in **Srivastava (2008)** and **Kumar (2018)**.

The set of bicomplex numbers is defined as:

$$\mathbb{C}_2 = \{x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4 : x_1, x_2, x_3, x_4 \in \mathbb{C}_0\}$$

where $i_1 \neq i_2$ , $i_1^2 = i_2^2 = -1$ and $i_1i_2 = i_2i_1$.

**Singular and Non-Singular Elements in $C_2$:**

The Bicomplex number $\xi = z_1 + i_2z_2 = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2 = {}^1\xi e_1 + {}^2\xi e_2$ is singular if and only if $z_1^2 + z_2^2 = 0$ or $(z_1 - i_1z_2 = 0$ or $z_1 + i_1z_2 = 0)$ or $({}^1\xi = 0$ or ${}^2\xi = 0)$ and is non-singular if and only if $z_1^2 + z_2^2 \neq 0$ or $(z_1 - i_1z_2 \neq 0$ and $z_1 + i_1z_2 \neq 0)$ or $({}^1\xi \neq 0$ and ${}^2\xi \neq 0)$. The set of all singular elements in $\mathbb{C}_2$ is denoted as $\mathbb{O}_2$ and $\mathbb{C}_2/\mathbb{O}_2$ is the set of all non-singular elements in $\mathbb{C}_2$.

We shall use the notations $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$ for the following sets:

$\mathbb{C}(i_1) = \{x + i_1y : x, y \in \mathbb{C}_0\}$ and $\mathbb{C}(i_2) = \{x + i_2y : x, y \in \mathbb{C}_0\}$.

The set of Hyperbolic Numbers is defined as $\mathbb{H} = \{x + i_1i_2y : x, y \in \mathbb{C}_0\}$.

**2. Bicomplex Polynomial**

A bicomplex polynomial is a function of the form

$$P(\xi) = \sum_{k=0}^{n} \gamma_k \xi^k$$

where $\gamma_k \in \mathbb{C}_2, k = 0,1,2, \dots, n$ and $\xi$ is a bicomplex variable.

The $\mathbb{C}(i_1)$ idempotent representation of bicomplex polynomial is given as

$$P(\xi) = \sum_{k=0}^{n} \gamma_k \xi^k = \left(\sum_{k=0}^{n} {}^1\gamma_k\, {}^1\xi^k\right) e_1 + \left(\sum_{k=0}^{n} {}^2\gamma_k\, {}^2\xi^k\right) e_2 = {}^1P({}^1\xi)e_1 + {}^2P({}^2\xi)e_2$$

${}^1P({}^1\xi) = \sum_{k=0}^{n} {}^1\gamma_k\, {}^1\xi^k$ and ${}^2P({}^2\xi) = \sum_{k=0}^{n} {}^2\gamma_k\, {}^2\xi^k$ are polynomial in $\mathbb{C}(i_1)$

where ${}^1\gamma_k$ and ${}^2\gamma_k \in \mathbb{C}(i_1), k = 0,1,2, \dots, n$ and ${}^1\xi, {}^2\xi$ are variables in $\mathbb{C}(i_1)$.

**2.1 Roots of Bicomplex Polynomial:**

The roots of a bicomplex polynomial have been investigated by **Pogorui et al. (2006)** and **Luna-Elizarrarás et al. (2015)**. We have also investigated the roots of a bicomplex polynomial from various perspectives.

Consider the bicomplex polynomial

$$P(\xi) = \sum_{k=0}^{n} \gamma_k \xi^k = \left(\sum_{k=0}^{n} {}^1\gamma_k\, {}^1\xi^k\right) e_1 + \left(\sum_{k=0}^{n} {}^2\gamma_k\, {}^2\xi^k\right) e_2 = {}^1P({}^1\xi)e_1 + {}^2P({}^2\xi)e_2$$

Then the roots of $P(\xi)$ were determined as follows:

**Case (1)** If ${}^1P({}^1\xi)$ is a polynomial of degree $k$, $1 \le k \le n$ and ${}^2P({}^2\xi)$ is a polynomial of degree $l$, $1 \le l \le n$ then $P(\xi)$ consist $kl$ roots counting multiplicities.

**Case (2)** $P(\xi)$ has infinite roots if

**(i)** ${}^1P({}^1\xi)$(respectively ${}^2P({}^2\xi)$ ) is a polynomial of degree $k$, $1 \le k \le n$ and ${}^2P({}^2\xi)$(respectively ${}^1P({}^1\xi)$ ) is a zero polynomial.

**(ii)** Both ${}^1P({}^1\xi)$ and ${}^2P({}^2\xi)$ are zero polynomials.

**Case (3)** $P(\xi)$ has no root if

**(i)** ${}^1P({}^1\xi)$(respectively ${}^2P({}^2\xi)$ ) is a polynomial of degree $k$, $0 \le k \le n$ and ${}^2P({}^2\xi)$(respectively ${}^1P({}^1\xi)$ ) is a constant polynomial.

**(ii)** ${}^1P({}^1\xi)$(respectively ${}^2P({}^2\xi)$ ) is a zero polynomial and ${}^2P({}^2\xi)$(respectively ${}^1P({}^1\xi)$ ) is a constant polynomial.

If $\gamma_n \notin \mathbb{O}_2$, the bicomplex polynomial can be determined by its roots i.e. the polynomial can be splits into linear factors. Therefore, we have considered $\gamma_n \notin \mathbb{O}_2$ to prove the following theorem.

**Theorem 2. 1**: The bicomplex polynomial $P(\xi) = \sum_{k=0}^{n} \gamma_k \xi^k, \gamma_n \notin \mathbb{O}_2$ has $n^2$ roots counting multiplicities.

**Proof:** Given $\gamma_n \notin \mathbb{O}_2$

$\gamma_n \notin \mathbb{O}_2 \Leftrightarrow {}^1\gamma_n \neq 0$ and ${}^2\gamma_n \neq 0$

$\Rightarrow {}^1P({}^1\xi) = \sum_{k=0}^{n} {}^1\gamma_k \, {}^1\xi^k$ and ${}^2P({}^2\xi) = \sum_{k=0}^{n} {}^2\gamma_k \, {}^2\xi^k$ both are polynomials of degree $n$ in $\mathbb{C}(i_1)$

$\Rightarrow {}^1P({}^1\xi)$ *and* ${}^2P({}^2\xi)$ *both consist $n$ zeros counting multiplicities*

Let ${}^1A = \{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n\}$ and ${}^2A = \{\beta_1, \beta_2, \beta_3, \dots, \beta_n\}$ are the multiset of $n$ roots of ${}^1P({}^1\xi)$ *and* ${}^2P({}^2\xi)$ respectively.

Then the multiset

$A = {}^1A\, e_1 + {}^2A\, e_2 = \{\alpha_i e_1 + \beta_j e_2 : i, j = 1,2,3, \dots, n\}$ is the collection of all $n^2$ roots of $P(\xi)$.

**Note 2.1: Kumar (2022)** has divided $n^2$ eigenvalues of a bicomplex matrix $A \in \mathbb{C}_2^{n \times n}$ in $n$ spectrums. In an analogous way, we divided the $n^2$ roots of a bicomplex polynomial

$$P(\xi) = \sum_{k=0}^{n} \gamma_k \xi^k, \gamma_n \notin \mathbb{O}_2$$

into $n$ multisets for which the polynomial can be splits into linear factors.

*Consider the bicomplex polynomial* $P(\xi) = \sum_{k=0}^{n} \gamma_k \xi^k, \gamma_n \notin \mathbb{O}_2$ .

Then, the multiset ${}^1A = \{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n\}$ and ${}^2A = \{\beta_1, \beta_2, \beta_3, \dots, \beta_n\}$ are roots of ${}^1P({}^1\xi)$ *and* ${}^2P({}^2\xi)$ respectively and the multiset $A = {}^1A\, e_1 + {}^2A\, e_2 = \{\alpha_i e_1 + \beta_j e_2 : i, j = 1,2,3, \dots, n\}$ is the set of all $n^2$ roots of $P(\xi)$.

We divide these $n^2$ roots of $P(\xi)$ in $n$ multisets $S_1, S_2, S_3, \dots, S_n$ in view the following fact

$$\begin{aligned} P(\xi) &= {}^1P({}^1\xi)e_1 + {}^2P({}^2\xi)e_2 \\ &= {}^1\gamma_n ({}^1\xi - \alpha_1)({}^1\xi - \alpha_2)({}^1\xi - \alpha_3) \dots ({}^1\xi - \alpha_{n-1})({}^1\xi - \alpha_n)\, e_1 \\ &\quad + {}^2\gamma_n ({}^2\xi - \beta_1)({}^2\xi - \beta_2)({}^2\xi - \beta_3) \dots ({}^2\xi - \beta_{n-1})({}^2\xi - \beta_n)\, e_2 \end{aligned}$$

**1.** $S_1 = \{\alpha_1 e_1 + \beta_1 e_2, \alpha_2 e_1 + \beta_2 e_2, \alpha_3 e_1 + \beta_3 e_2, \dots \dots \dots, \alpha_{n-1} e_1 + \beta_{n-1} e_2, \alpha_n e_1 + \beta_n e_2\}$

$$\begin{aligned} P(\xi) = \gamma_n &\{\xi - (\alpha_1 e_1 + \beta_1 e_2)\}\{\xi - (\alpha_2 e_1 + \beta_2 e_2)\}\{\xi - (\alpha_3 e_1 + \beta_3 e_2)\} \dots \dots \dots \\ &\dots \dots \dots \{\xi - (\alpha_{n-1} e_1 + \beta_{n-1} e_2)\}\{\xi - (\alpha_n e_1 + \beta_n e_2)\} \end{aligned}$$

**2.** $S_2 = \{\alpha_1 e_1 + \beta_2 e_2, \alpha_2 e_1 + \beta_3 e_2, \alpha_3 e_1 + \beta_4 e_2, \dots \dots \dots, \alpha_{n-1} e_1 + \beta_n e_2, \alpha_n e_1 + \beta_1 e_2\}$

$$\begin{aligned} P(\xi) = \gamma_n &\{\xi - (\alpha_1 e_1 + \beta_2 e_2)\}\{\xi - (\alpha_2 e_1 + \beta_3 e_2)\}\{\xi - (\alpha_3 e_1 + \beta_4 e_2)\} \dots \dots \dots \\ &\dots \dots \dots \{\xi - (\alpha_{n-1} e_1 + \beta_n e_2)\}\{\xi - (\alpha_n e_1 + \beta_1 e_2)\} \end{aligned}$$

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**3.** $S_3 = \{\alpha_1 e_1 + \beta_3 e_2, \alpha_2 e_1 + \beta_4 e_2, \alpha_3 e_1 + \beta_5 e_2, \dots \dots \dots \dots, \alpha_{n-1} e_1 + \beta_1 e_2, \alpha_n e_1 + \beta_2 e_2\}$

$P(\xi) = \gamma_n \{\xi - (\alpha_1 e_1 + \beta_3 e_2)\}\{\xi - (\alpha_2 e_1 + \beta_4 e_2)\}\{\xi - (\alpha_3 e_1 + \beta_5 e_2)\} \dots \dots \dots \dots$

$\dots \dots \dots \dots \{\xi - (\alpha_{n-1} e_1 + \beta_1 e_2)\}\{\xi - (\alpha_n e_1 + \beta_2 e_2)\}$

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**(n-1).** $S_{n-1} = \{\alpha_1 e_1 + \beta_{n-1} e_2, \alpha_2 e_1 + \beta_n e_2, \alpha_3 e_1 + \beta_1 e_2, \dots \dots \dots \dots, \alpha_{n-1} e_1 + \beta_{n-3} e_2, \alpha_n e_1 + \beta_{n-2} e_2\}$

$P(\xi) = \gamma_n \{\xi - (\alpha_1 e_1 + \beta_{n-1} e_2)\}\{\xi - (\alpha_2 e_1 + \beta_n e_2)\}\{\xi - (\alpha_3 e_1 + \beta_1 e_2)\} \dots \dots \dots \dots$

$\dots \dots \dots \dots \{\xi - (\alpha_{n-1} e_1 + \beta_{n-3} e_2)\}\{\xi - (\alpha_n e_1 + \beta_{n-2} e_2)\}$

**(n).** $S_n = \{\alpha_1 e_1 + \beta_n e_2, \alpha_2 e_1 + \beta_1 e_2, \alpha_3 e_1 + \beta_2 e_2, \dots \dots \dots \dots, \alpha_{n-1} e_1 + \beta_{n-2} e_2, \alpha_n e_1 + \beta_{n-1} e_2\}$

$P(\xi) = \gamma_n \{\xi - (\alpha_1 e_1 + \beta_n e_2)\}\{\xi - (\alpha_2 e_1 + \beta_1 e_2)\}\{\xi - (\alpha_3 e_1 + \beta_2 e_2)\} \dots \dots \dots \dots$

$\dots \dots \dots \dots \{\xi - (\alpha_{n-1} e_1 + \beta_{n-2} e_2)\}\{\xi - (\alpha_n e_1 + \beta_{n-1} e_2)\}$

The solutions $\xi \in \mathbb{C}_2$ of $\xi^2 = \eta$ for various values of $\eta$ are provided by the **theorems 2.2** to **2.7**.

**Theorem 2.2:** For a given $r \in \mathbb{C}_0$, the solutions of $\xi^2 = r$, are given by $\xi = \pm\sqrt{r}, \pm i_1 i_2 \sqrt{r}$.

**Proof:** Let $\xi^2 = r$

$\Rightarrow {}^1\xi^2\, e_1 + {}^2\xi^2\, e_2 = r e_1 + r e_2$

$\Rightarrow {}^1\xi^2 = r$ and ${}^2\xi^2 = r$

$\Rightarrow {}^1\xi = \pm\sqrt{r}$ and ${}^2\xi = \pm\sqrt{r}$

$\Rightarrow \xi = {}^1\xi e_1 + {}^2\xi e_2 = \pm\sqrt{r}, \pm i_1 i_2 \sqrt{r}$

**Theorem 2.3:** For a given $+i_1 s \in \mathbb{C}(i_1), s \neq 0$ , the solutions of $\xi^2 = r + i_1 s$ are given by

$$\xi = \pm \frac{\sqrt{|r + i_1 s|}(r + i_1 s + |r + i_1 s|)}{|r + i_1 s + |r + i_1 s||}, \pm i_1 i_2 \frac{\sqrt{|r + i_1 s|}(r + i_1 s + |r + i_1 s|)}{|r + i_1 s + |r + i_1 s||}$$

**Proof:**

$\xi^2 = r + i_1 s$

$\Rightarrow ({}^1\xi e_1 + {}^2\xi e_2)^2 = (r + i_1 s) e_1 + (r + i_1 s) e_2$

$\Rightarrow {}^1\xi^2 e_1 + {}^2\xi^2 e_2 = (r + i_1 s) e_1 + (r + i_1 s) e_2$

$\Rightarrow {}^1\xi^2 = r + i_1 s$ and ${}^2\xi^2 = r + i_1 s$

As, $s \neq 0$

$${}^1\xi = \pm \frac{\sqrt{|r + i_1 s|}(r + i_1 s + |r + i_1 s|)}{|r + i_1 s + |r + i_1 s||} \quad and \quad {}^2\xi = \pm \frac{\sqrt{|r + i_1 s|}(r + i_1 s + |r + i_1 s|)}{|r + i_1 s + |r + i_1 s||}$$

$\Rightarrow \xi = {}^1\xi e_1 + {}^2\xi e_2$

$$= \pm \frac{\sqrt{|r + i_1 s|}(r + i_1 s + |r + i_1 s|)}{|r + i_1 s + |r + i_1 s||}, \pm i_1 i_2 \frac{\sqrt{|r + i_1 s|}(r + i_1 s + |r + i_1 s|)}{|r + i_1 s + |r + i_1 s||}$$

**Theorem 2.4:** For a given $r + i_2 s \in \mathbb{C}(i_2), s \neq 0$, the solutions of $\xi^2 = r + i_2 s$ are given by

$$\xi = \pm \frac{\sqrt{|r + i_2 s|}(r + i_2 s + |r + i_2 s|)}{|r + i_2 s + |r + i_2 s||}, \pm i_1 i_2 \frac{\sqrt{|r + i_2 s|}(r + i_2 s + |r + i_2 s|)}{|r + i_2 s + |r + i_2 s||}$$

**Proof:** The proof can easily be seen by considering $\mathbb{C}(i_2)$ −idempotent representation.

**Theorem 2.5:** For a given $a e_1 + b e_2 \in \mathbb{H}$, the solutions of $\xi^2 = a e_1 + b e_2$ are given by

$$\xi = \pm(\sqrt{a}\, e_1 + \sqrt{b} e_2), \pm i_1 i_2 (\sqrt{a}\, e_1 + \sqrt{b} e_2).$$

**Proof:**

$\xi^2 = a e_1 + b e_2$

$\Rightarrow ({}^1\xi e_1 + {}^2\xi e_2)^2 = a e_1 + b e_2$

$\Rightarrow {}^1\xi^2 e_1 + {}^2\xi^2 e_2 = a e_1 + b e_2$

$\Longrightarrow {}^1\xi^2 = a$ and ${}^2\xi^2 = b$

$\Longrightarrow {}^1\xi = \pm\sqrt{a}$ and ${}^2\xi = \pm\sqrt{b}$

$\Longrightarrow \xi = {}^1\xi e_1 + {}^2\xi e_2$

$$= \pm(\sqrt{a}\, e_1 + \sqrt{b} e_2), \pm i_1 i_2(\sqrt{a}\, e_1 + \sqrt{b} e_2)$$

**Theorem 2.6:** For a given $\eta = ze_1 + we_2 \in \mathbb{C}_2;\ z = a + i_1 b, w = c + i_1 d \in \mathbb{C}(i_1)$ the solutions of $\xi^2 = \eta$ are given by

$$\xi = \pm\frac{1}{\sqrt{2}}\left[\left(\sqrt{|z| + a} + i_1 sgn(b)\sqrt{|z| - a}\right) e_1 + \left(\sqrt{|w| + c} + i_1 sgn(d)\sqrt{|w| - c}\right) e_2\right],$$

$$\pm\frac{i_1 i_2}{\sqrt{2}}\left[\left(\sqrt{|z| + a} + i_1 sgn(b)\sqrt{|z| - a}\right) e_1 + \left(\sqrt{|w| + c} + i_1 sgn(d)\sqrt{|w| - c}\right) e_2\right].$$

where, $sgn(b) = \begin{cases} +1, if\ b \geq 0 \\ -1, if\ b < 0 \end{cases}$ and $sgn(d) = \begin{cases} +1, if\ d \geq 0 \\ -1, if\ d < 0 \end{cases}$

**Proof:** Let $\xi = ue_1 + ve_2 \in C_2;\ u, v \in \mathbb{C}(i_1)$

$\xi^2 = \eta$

$\Rightarrow (ue_1 + ve_2)^2 = ze_1 + we_2$

$\Rightarrow u^2 e_1 + v^2 e_2 = ze_1 + we_2$

$\Rightarrow u^2 = z$ and $v^2 = w$

$\Rightarrow u = \pm\frac{1}{\sqrt{2}}\left[\sqrt{|z| + a} + i_1 sgn(b)\sqrt{|z| - a}\right]$ and $v = \pm\frac{1}{\sqrt{2}}\left[\sqrt{|w| + c} + i_1 sgn(d)\sqrt{|w| - c}\right]$

$\Rightarrow \xi = ue_1 + ve_2$

$$= \pm\frac{1}{\sqrt{2}}\left[\left(\sqrt{|z| + a} + i_1 sgn(b)\sqrt{|z| - a}\right) e_1 + \left(\sqrt{|w| + c} + i_1 sgn(d)\sqrt{|w| - c}\right) e_2\right],$$

$$\pm\frac{i_1 i_2}{\sqrt{2}}\left[\left(\sqrt{|z| + a} + i_1 sgn(b)\sqrt{|z| - a}\right) e_1 + \left(\sqrt{|w| + c} + i_1 sgn(d)\sqrt{|w| - c}\right) e_2\right].$$

**Corollary 2.1:** For a given $\eta = ze_1 + we_2 \in \mathbb{C}_2;\ z = a + i_1 b, w = c + i_1 d \in \mathbb{C}(i_1), b \neq 0, d \neq 0$, then the solutions of $\xi^2 = \eta$ are given by

$$\xi = \pm\frac{1}{\sqrt{2}}\left[\left(\sqrt{|z| + a} + i_1 \frac{b}{|b|}\sqrt{|z| - a}\right) e_1 + \left(\sqrt{|w| + c} + i_1 \frac{d}{|d|}\sqrt{|w| - c}\right) e_2\right],$$

$$\pm\frac{i_1 i_2}{\sqrt{2}}\left[\left(\sqrt{|z| + a} + i_1 \frac{b}{|b|}\sqrt{|z| - a}\right) e_1 + \left(\sqrt{|w| + c} + i_1 \frac{d}{|d|}\sqrt{|w| - c}\right) e_2\right].$$

**Theorem 2.7:** For a given $\eta = ze_1 + we_2 \in \mathbb{C}_2;\ z, w \in \mathbb{C}(i_1), I(z) \neq 0, I(w) \neq 0$, the solutions of $\xi^2 = \eta$ are given by

$$\xi = \pm\left(\frac{\sqrt{|z|}(z + |z|)}{|z + |z||}\, e_1 + \frac{\sqrt{|w|}(w + |w|)}{|w + |w||}\, e_2\right), \pm i_1 i_2\left(\frac{\sqrt{|z|}(z + |z|)}{|z + |z||}\, e_1 + \frac{\sqrt{|w|}(w + |w|)}{|w + |w||}\, e_2\right)$$

**Proof:** Let $\xi = ue_1 + ve_2 \in \mathbb{C}_2;\ u, v \in \mathbb{C}(i_1)$

$\xi^2 = \eta$

$\Rightarrow (ue_1 + ve_2)^2 = ze_1 + we_2$

$\Rightarrow u^2 e_1 + v^2 e_2 = ze_1 + we_2$

$\Rightarrow u^2 = z$ and $v^2 = w$

$$As, I(z) \neq 0\ and\ I(w) \neq 0, u = \pm\frac{\sqrt{|z|}(z + |z|)}{|z + |z||}\ and\ v = \pm\frac{\sqrt{|w|}(w + |w|)}{|w + |w||}$$

$\Rightarrow\ \xi = ue_1 + ve_2$

$$= \pm\left(\frac{\sqrt{|z|}(z + |z|)}{|z + |z||}\, e_1 + \frac{\sqrt{|w|}(w + |w|)}{|w + |w||}\, e_2\right),$$

$\pm i_1 i_2 \left( \frac{\sqrt{|z|}(z+|z|)}{|z+|z||} e_1 + \frac{\sqrt{|w|}(w+|w|)}{|w+|w||} e_2 \right).$

**2.2. The nth roots of unity in $\mathbb{C}_2$**

**Theorem 2.8:** The set of all roots of $\xi^n = 1$, $G = \{\omega^r e_1 + \omega^s e_2 : r, s = 0,1,2, \ldots n-1\}, where\ \omega = e^{2\pi i_1/n}$ form an abelian group under multiplication.

**Proof:** Let $\xi^n = 1$

$$\Rightarrow ({}^1\xi e_1 + {}^2\xi e_2)^n = e_1 + e_2$$
$$\Rightarrow {}^1\xi^n e_1 + {}^2\xi^n e_2 = e_1 + e_2$$

$\Rightarrow {}^1\xi^n = 1$ and ${}^2\xi^n = 1$

$\Rightarrow {}^1\xi = e^{2r\pi i_1/n}, r = 0,1,2, \ldots\ldots n-1$ and ${}^2\xi = e^{2s\pi i_1/n}, s = 0,1,2, \ldots\ldots n-1$

$$Let\ \omega = e^{2\pi i_1/n}$$

$\Rightarrow {}^1\xi = \omega^r, r = 0,1,2, \ldots\ldots n-1$ and ${}^2\xi = \omega^s, s = 0,1,2, \ldots\ldots n-1$

$\Rightarrow \xi = {}^1\xi e_1 + {}^2\xi e_2 = \omega^r e_1 + \omega^s e_2 : r, s = 0,1,2, \ldots\ldots n-1$

Let $G = \{\omega^r e_1 + \omega^s e_2 : r, s = 0,1,2, \ldots\ldots n-1\}$ be a collection of all these $n^2$ roots.

Since $C_n = \{1, \omega, \omega^2, \ldots, \omega^{n-1}\}$ is a cyclic group of order $n$ in $\mathbb{C}(i_1)$, therefore the direct product $G = C_n \times C_n = \{\omega^r e_1 + \omega^s e_2 : r, s = 0,1,2, \ldots n-1\}$ is an abelian group of order $n^2$ in $\mathbb{C}_2$.

It has been attempted to characterize distinct cyclic subgroups within an abelian group of the nth root of unity. Here we have discussed two groups of order $2^2$ and $3^2$.

**1.** The solutions of $\xi^2 = 1$ are given by $1, -1, i_1 i_2, -i_1 i_2$. The set of all roots of $\xi^2 = 1$, $G = \{1, -1, i_1 i_2, -i_1 i_2\}$ form an abelian group under multiplication.

There are four distinct cyclic subgroups of $G$ from which one is trivial and other three are of order 2.

$H_1 = \{1\}, H_2 = \{1, -1\}, H_3 = \{1, i_1 i_2\}, H_4 = \{1, -i_1 i_2\}$

| $\lvert G\rvert$ | **Elements of $G$** | | **Cyclic Subgroup of $G$** | | $\lvert c(G)\rvert$ |
|---|---|---|---|---|---|
| | Order $n$ | No. of Elements of order n | Order $n$ | No. of Cyclic Subgroup of order $n$ | |
| $2^2$ | 1 | 1 | 1 | 1 | 4 |
| | 2 | 3 | 2 | 3 | |
| $\lvert G\rvert$ | | 4 | $\lvert c(G)\rvert$ | 4 | |

**2.** The set of all roots of $\xi^3 = 1$, $G = \{1, e_1 + \omega e_2, e_1 + \omega^2 e_2, \omega e_1 + e_2, \omega, \omega e_1 + \omega^2 e_2, \omega^2 e_1 + e_2, \omega^2 e_1 + \omega e_2, \omega^2\}$, where $\omega = e^{2\pi i_1/3}$form an abelian group under multiplication.

There are five distinct cyclic subgroup of $G$ from which one is trivial and other four are of order 3.

$$H_1 = \{1\}$$

$H_2 = \{1, \omega, \omega^2\}$

$H_3 = \{1, e_1 + \omega e_2, e_1 + \omega^2 e_2\}$

$H_4 = \{1, \omega e_1 + e_2, \omega^2 e_1 + e_2\}$

$H_5 = \{1, \omega e_1 + \omega^2 e_2, \omega^2 e_1 + \omega e_2\}$

| $\lvert G\rvert$ | **Elements of $G$** | | **Cyclic Subgroup of $G$** | | $\lvert c(G)\rvert$ |
|---|---|---|---|---|---|
| | Order $n$ | No. of Elements of order $n$ | Order N | No. of Cyclic Subgroup of order $n$ | |
| $3^2$ | 1 | 1 | 1 | 1 | 5 |
| | 3 | 8 | 3 | 4 | |
| | $\lvert G\rvert$ | 9 | $\lvert c(G)\rvert$ | 5 | |

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**3**. The set of all roots of $\xi^n = 1$, $G = \{\omega^r e_1 + \omega^s e_2 : r, s = 0,1,2, \ldots.. n-1\}, where\ \omega = e^{2\pi i_1/n}$ form an abelian group under multiplication.

**(i)** No. of distinct cyclic subgroup of an abelian group of order $p^{2n}$, where $p$ is prime and $n \in \mathbb{N}$ is given by $\frac{1}{p-1}[p^{n+1} + p^n - 2]$

**(ii)** No. of distinct cyclic subgroup of an abelian group of order $p_1^{2n_1} p_2^{2n_2} \ldots \ldots p_k^{2n_k}$, where $p_1, p_2, \ldots \ldots, p_k$ are primes and $n_1, n_2, \ldots \ldots n_k \in \mathbb{N}$ is given by

$$\frac{1}{p_1 - 1}[p_1^{n_1+1} + p_1^{n_1} - 2] \times \frac{1}{p_2 - 1}[p_2^{n_2+1} + p_2^{n_2} - 2] \times \ldots \ldots \ldots \ldots \ldots \ldots \times \frac{1}{p_k - 1}[p_k^{n_k+1} + p_k^{n_k} - 2]$$

## II. CONCLUSION

In recent years, mathematicians and physicists have more interest of their study on bicomplex algebras. In this article we have attempted to investigate the roots of bicomplex polynomial. A polynomial of degree n with non-singular leading coefficient have at most $n^2$ roots and n multiset of roots for which the polynomial can be splits into linear factors. But a polynomial with singular leading coefficient which have finite roots cannot be split into linear factors. Square roots for several quadratic equations using bicomplex variables have been obtained. Within an abelian group of $n^2$nth roots of unity in $\mathbb{C}_2$, a general formula for counting the number of distinct cyclic subgroups is provided.

## III. REFERENCES

[1] Price, G. B. (1991). An introduction to multicomplex space and Functions. Marcel Dekker

[2] Luna-Elizarrarás, M.E., Shapiro, M., Struppa, D. C., Vajiac, A.(2015). Bicomplex Holomorphic Functions:The Algebra, Geometry and Analysis of Bicomplex Numbers. Springer International Publishing.

[3] Kumar, J.(2018). On Some Properties of Bicomplex Numbers •Conjugates •Inverse •Modulii. Journal of Emerging Technologies and Innovative Research. 5(9):475-499

[4] Srivastava, Rajiv K. (2008). Certain Topological Aspects of Bicomplex Space. Bull. Pure & Appl. Math. 222-234.

[5] Pogorui, A.A. & Rodriguez-Dagnino, R.M. (2006). On the set of zeros of bicomplex polynomials. Complex Variables and Elliptic Equations. 51(7):725–730.

[6] Kumar,J.(2022). Diagonalization of the Bicomplex Matrix. Int. J. Of Research Publication and Reviews. 3(11): 105-116.