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A STUDY OF THE FIXED POINT MEASURABLE MULTIFUNCTION THEOREM IN DIFFERENT SPACES AND ITS APPLICATIONS

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ABSTRACT

The study of a wide range of equation types that arise in the fields of engineering, biology, physics, and other sciences and technologies has demonstrated the essential importance of fixed point theory. A fixed point of P is a point x of X for which P(x) = x for a function P that has a set X as both domain and range. The two basic theorems pertaining to fixed points are the Banach and Brouwer theorems. The distinction between the two main subfields of fixed point theory—metric fixed point theory and topological fixed point theory—is demonstrated by the Banach and Brouwer theorems. Caccioppoli started the systematic application of the Banachs principal to several existence theorems in analysis in 1930. Following that, this principle evolved into a simple instrument for resolving a variety of physics and economics issues. This principle has since been expanded and extended to a variety of metric and generalized metric spaces.

Keywords: Fixed Point Theorems, Mathematics, Mathematics Interdisciplinary Applications Metric Spaces, Physical Sciences.

I. INTRODUCTION

We introduce some basic fixed point theory notions and ideas at the beginning of this paper. After introducing the concept of metric space, we define a number of contraction and associated fixed point theorems. The definitions of reciprocal continuity of functions, compatibility, weak compatibility, commutativity, R-weak commutativity, and weak compatibility come next. Metric spaces and the E.A. property. Due to its wide range of applications in both pure and applied mathematics as well as other disciplines like economics, biology, and physical science, the theory of fixed points is currently a promising area of mathematics, particularly in nonlinear functional analysis. Fixed point theory is a rapidly growing topic that has arisen as a technique for solving various nonlinear differential and integral equations. The goal of the abstract mathematical field of functional analysis is to create a mathematical language. It is derived from classical analysis. Generally, problems from different disciplines share a number of traits and attributes. This information was utilized to develop a method that effectively solves problems by omitting unnecessary constraints. It is feasible to highlight only the most crucial elements and direct the investigator's attention toward the most crucial elements by disregarding unimportant aspects, which is what is intended. It is crucial to use an abstract approach that can handle mathematical systems with ease because to its straightforward and effective methods. The concepts and techniques of functional analysis have proven essential to a variety of mathematical fields and their uses since they were first introduced over 80 years ago. The theories of linear algebra, linear ordinary and partial differential equations, calculus of variations, approximation theory, and linear integral equations served as the basis for the creation and dissemination of contemporary concepts. Functional analysis methods address both linear and nonlinear issues. In addition to functional analytic methods, linear algebraic methods are more frequently employed to tackle linear problems. However, fixed point theory is the essential instrument for resolving non-linear issues. A fixed point of the function is a point of the domain that is mapped onto itself. The center of a wheel, for instance, is a fixed point since we observe that all of its points move when we watch it rotate. The point at which equilibrium or steadystate conditions are achieved in real-world scenarios is known as a fixed point. Water's boiling point is a temperature that never changes. A substance's triple point and the temperature at which water freezes are both fixed points. Fixed points are typically used to explain the ideas of stability and equilibrium. In problem-solving strategies, the existence of at least one fixed point under certain general conditions is guaranteed by theorems. Thus, one of the most active fields of study in functional analysis is fixed point theory. Chemistry, economics, engineering, game theory, computer science, physics, geometry, astronomy, fluid and elastic mechanics, control theory, image processing, and biology are just a few of the fields in which it finds use. While fixed point theory addresses several issues such as fixed point computation methods, algorithms, and approximation approaches, its primary focus is on predicting the existence of fixed points. Fixed point theory is used to solve problems involving split feasibility, variation inequality, nonlinear optimization, equilibrium, complementarity, selection and matching, and problems proving the existence of solutions to differential and integral equations.

Historical Background of Fixed Point Theory Fixed point theory has been widely applied in many areas of mathematics and mathematical sciences since its birth, encouraging researchers to make new discoveries. Fixed point theory now has a new angle thanks to computer technology. Additionally, the development of new software cleared the path for rapid and effective computations. The development of fixed point theory has followed a lengthy, wide-ranging, and complex path. Henri Poincare's work in the 1880s marked the beginning of the development of fixed point outcomes [81]. One of Poincare's most significant and inventive contributions to the development of fixed point theory was his work on topological or qualitative approaches in the study of nonlinear issues in analysis. However, in 1912, Brouwer presented the most significant fixed point result in history. The fixed point theorem was established by Brouwer as the answer to the equation f(x) = x. It asserts that there is a point x such that f(x) = x for any continuous function f mapping a compact convex set to itself. Brouwer's ongoing work established fixed point theorems for squares, spheres, and their n-dimensional equivalents, which Kakutani [54] further expanded. However, because many situations involve infinite-dimensional spaces, the Brouwer fixed point theorem only considers the case of finite-dimensional space, which is not very useful. In 1922, Birkhoff and Kellogg [20] became the first mathematicians to investigate infinite-dimensional fixed point theorems. Under the presumptions of convexity and compactness, they provided a proof for Brouwer's fixed point theorem. For the situation of metric linear space, Schauder [56] extended the Birkhoff-Kellogg theorem later in 1927. The Brouwer fixed point result, which asserts that each compact convex set in a Banach space has the fixed point attribute, was also expanded by him in 1930. In the meantime, the Banach contraction principle was created in 1922 and is thought to be the foundation of fixed point theory. In the field of a complete metric space, Banach [14] proved that a contraction mapping has a single

fixed point. Because the contraction mapping in a complete metric space yields a fixed point, which is the precise solution, it is highly helpful in existential and uniqueness theories. Rhoades (1977), Caristi (1975), Hussain and Sehgal (1975), and Kannan (1968) all went on to work on the Banach contraction principle and provide additional findings. For decades, the Banach contraction principle and the theory of fixed points have been studied and applied in a variety of fields, yielding numerous fixed point conclusions. Jungck's [51] fixed point theorem on commutative mappings was introduced, while Sessa's [37] later relaxed the commutativity restriction. findings and associated ideas, fixed point theory's development takes a new direction. Ciric's work was followed by Rhoades [85] and Kirk [59] on non-expansive mappings, while Park [79] and Sadovski [90] added significantly with their work on new mapping conditions. Subsequently, Mann [67] and Ishikawa [48] developed a method for finding fixed points that utilized the approximate limit generated by iterative sequence convergence. When the equations do not produce a precise solution, such an approximation response is desired. This influenced the transition of fixed point theory into a new age. It is clear from examining the development of fixed point theory that the fixed point is reached by either changing the mapping properties or implementing modifications to the space's structure. Only a basic overview of the history of fixed point theory development is provided here; in-depth analyses of the subject may be found in other linked articles as well as in the works of Chetan and Vandna [25], Almezel et al. [6], and Pant et al. [78].

Theorem 1 : Let $(X, \tau 1)$ & $(Y, \tau 2)$ be sf ts 's and $f : X \to Y$ be a mapping. Then

(i) f is f its mapping.

(ii) $f^{-1}(\mu)$ is r - fMo in X $\forall \mu \in I^Y$, $r \in I0$ with $\tau 2(1 - \mu) \geq r$.

(iii) $f(MI\tau 1(\lambda, r)) \leq C\tau 2(f(\lambda), r)$, $\forall \lambda \in I^X$ & $r \in I0$.

(iv) $MI\tau 1(f^{-1}(\mu), r) \leq f^{-1}(C\tau 2(\mu, r))$, $\forall \mu \in I^Y$ & $r \in I0$.

(v) Cτ1 (θIτ1 (f^{-1} (μ), r), r)∨Iτ1 (δCτ1 (f^{-1} (μ), r), r) ≥ f^{-1} (Cτ2 (μ, r)), ∀ μ ∈ I Y & r ∈ I .

(vi) f^{-1} (Iτ2 (μ, r)) ≥ MCτ1 (f^{-1} (μ), r), ∀ μ ∈ I Y & r ∈ I0.

(μ, Proof. (i)⇒ (ii): Let μ ∈ I Y , r ∈ I0 with τ2(1 − μ) ≥ r.

Since f is f mapping, f −1 (1 − μ) is an set of X .

But f^{-1} (1 − μ) = 1 − f^{-1} (μ).

Therefore f −1 (μ) is an r − f set of X . (ii)⇒ (iii):

By Theorem, we have, f^{-1} (1 − μ)

≥ MIτ1 (f^{-1} (1 − μ), r)

≥ 1 − (M (f^{-1} (μ), r)) f^{-1} (μ)

≥ MCτ1 (f^{-1} (μ), r).

From Theorem 1.2.2, we have f^{-1} (μ)

≤ M (f^{-1} (μ), r).

Thus, f^{-1} (μ) = MCτ1 (f^{-1} (μ), r).

Hence f^{-1} (μ) is r − f Mc set in X. (ii)⇒ (v): For all μ ∈ I Y , r ∈ I0 ,

Since τ2(1 − Cτ2 (μ, r)) ≥ r. Then by (ii), we see that f^{-1} (Cτ2 (μ, r)) is r − f Mo set in X.

Then by (v), Cτ1 (θIτ1 (f^{-1} (μ), r), r) ∨ Iτ1 (δCτ1 (f^{-1} (μ), r), r) ≥ f^{-1} (Cτ2 (μ, r)) = f^{-1} (μ).

Hence f^{-1} (μ) is r - f M_0 set in X. (iv)⇒ (vi): It is easily proved from Theorem . (vi)⇒ (i): Let μ be r - fo set of Y. Then μ = (μ, r)

By (vi), f^{-1} (μ) ≥ 1 (f^{-1} (μ), r).

On the other hand, by Theorem f^{-1} (μ) ≤ M (f^{-1} (μ), r).

Thus, f^{-1} (μ) = M (f^{-1} (μ), r), that is, f^{-1} (μ)

Example 2: Assuming X = R and f(x) = 1, we may determine whether X is rational.

= 0, f is not continuous and is not a contraction mapping if X is irrational. However,

f^2(x)={f(1)},

Reasonable F(0) is an example of an irrational contraction mapping. However, the fixed point is the same for P and f. Interpretation Assume that X is a metric space that is compact. If the following criteria are met, the function

h: X × X → [0, ∞] is said to have attribute P:

Then h(x , y) = 0 , if x = 0

$\mathrm{Lim}_{n\to\infty} x_n$, where $x_n = 0$

$\lim_{n\to\infty} y_n$, where $y_n = 0$ and

$\lim_{n\to\infty} h(x_n , y_n) = 0$

Proposition 3 : :x_n∈ A, y_n∈ B (n ∈ N);

$\|x_n - y_n\|$ → dist.(A, B) whenever $\|x_n - y_n\|$ → dist(A, B).

Proof. Suppose $\|x_n - y_n\|$ → dist.(A, B) where (x_n) in A and (y_n) in B. Then for ∈> 0 there exists δ > 0 such that $\|T_X - T_Y\|$ ≤ dist.(A, B) + ∈ whenever x ∈ A, y ∈ B and

$\|x - y\|$≤ dist.(A, B) + δ. Now there exists a n_0∈ Z + such that $\|x_n - y_n\|$≤ dist(A, B) + δ for all n ≥ n_0.

Hence $\|Tx_n - Ty_n\|$→ dist.(A, B) + ∈ for all n ≥ n_0. This shows that $\|Tx_n - T y_n\|$ → dist.(A, B).

Conversely, suppose

T(A) ⊆ B, T(B) ⊆ A and T : A ∪ B → A ∪ B is not a relatively u-continuous mapping.

Then there exists ∈ > 0 such that for every n ∈ Z + there are x_n in A and y_n in B satisfying $\|x_n - y_n\|$ ≤ dist.(A, B) + 1 n and

$\|Tx_n - Ty_n\|$ ≥ dist.(A, B) + ∈ <0.

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This shows that there exist (x_n) in A and (y_n) in B such that

$\|Tx_n - Ty_n\|$ dist(A, B) but $\|x_n - y_n\| \to$ dist(A, B).

Hence T is relatively u-continuous.

Metric Spaces in Fixed Point Theory : The structure that separates a space from a set is something that a mere collection of items does not have. A distance function is all that a metric is. In other words, metric space is essentially a set that has a structure established by a precise definition of distance. The definition of metric spaces was first presented by Maurice Fr´echet [38] in 1906:

Definition 4. ([35]) Let X be a set. A function $d : X \times X \to R$ is said to be a metric if it satisfies the following properties:

(i)$d(x, y) = 0 \Leftrightarrow x = y$ (Indiscernible identity),

(ii) $d(x, y) = d(y, x)$ (Symmetry),

(iii)$d(x, y) \leq d(x, z) + d(z, y)$ (Triangle inequality).

Then the space X endowed with the metric d is called a metric space and is denoted by (X, d).

The most familiar metric is the Euclidean metric $(a, b) \to |a - b|$ in R. Trivially and uselessly the empty set φ is also a metric space. The modulus function $(z, w) \to |z - w|$ yields the Euclidean metric, which is the standard metric on the complex plane C. The shortest line segment that connects two points on a circle

Historical Flow of Generalized Metric Spaces : To obtain more general fixed point solutions, numerous attempts have been made to expand the notion of a metric space. Gahler [39] first proposed the idea of 2-metric spaces in 1963. Matthews [68] replaced the reflexivity property of general definition to create partial metric spaces later in 1992. Mustafa et al. [24] created G-metric space in 2006, while Huang et al. [25] presented the idea of cone metric spaces in 2007. Recently, several new kinds of generalized metric spaces have been introduced, and numerous spaces constructed as a cross between the previous varieties. The following is a summary of the historical flow of generalized forms of metric spaces that are hybrids of the earlier varieties. Sedghi et al. introduced D-metric space in 2007 [16]. 2009 saw the development of topological vector space valued cone metric spaces by Beg et al. [16] and cone rectangular metric spaces by Azam et al. [12].

Review of Related Works : Developing or addressing different metric spaces is the main problem while discussing fixed point theorems in different metric spaces. The previous section's analysis of the history of metric spaces makes clear how challenging it is to come up with novel combinations and ideas that differ from the ones that already exist. To do thus, a careful analysis of the current findings was carried out, and a few of them were chosen for additional research. This work focuses primarily on five types of metric spaces. The interval metric [10], vector-valued metric [5], Czerwik generalized metric [5], complex-valued fuzzy metric [10], and soft metric [10] are the metrics that are mainly studied. Every pertinent study is listed below. This section examines several fundamental definitions and characteristics of metric spaces, among other ideas that are essential to our later study.

Definition 4.1. ([11]) Let $a, b \in R$ be such that $a \leq b$. The set $\{x \in R ; a \leq x \leq b\}$ is called an interval and is denoted by IR.

Definition 5. Diameter of an interval is the length of the line segment joins the two extremes of the interval. For example, The diameter of the interval [3, 19] is 19 − 3 = 16.

Definition 6. ([10]) Let M be a set. A function $d : M \times M \to IR$ is said to be an interval metric if it satisfies the following properties:

(i) $0 \in d(x, x)$ (Reflexivity);

(ii) $d(x, y) = d(y, x)$ (Symmetry);

(iii) $|d(x, y)|M \leq |d(x, z)|M + |d(z, y)|M$ (Triangle inequality);

(iv) If $0 \in d(x, y) = d(x, x) = d(y, y)$, then $x = y$ (Indiscernible identity).

Definition 7.. ([6]) Let (X, d) be a complete metric space. $f : X \to X$ be a continuous function such that f is said to be a contraction if, $d(fx, fy) \leq \alpha\, d(x, y)$, $0 < \alpha < 1$, $x, y \in X$

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Definition 8.. Let X = C[a, b] be the function space. The interval metric in C[a, b], we denote it by dl, is defined in such a way that dl : X × X → IR as

Note 9.. In all the previous works on interval metric, the Moore interval module was used as a tool to prove the triangle inequality condition. So that the defined metric will always coincide with the maximum metric in C[a, b]. So the equivalence is trivial.

Examples 10: Let $\mathbb{M}$ = [0, 1], Define $\emptyset, \psi$: $\mathbb{M} \to \mathbb{M}$ such that

$\emptyset(u) = 2 - u$ and $\psi(u) = \frac{3-\mathrm{u}}{2}$ Consider a sequence

$up = \frac{1+1}{\mathrm{p}}$ we have, $\lim p{\to}\infty\ \emptyset(\mathrm{u_p})$

$= \frac{2-1}{1} p = 1$ and $\lim p{\to}\infty\ (\mathrm{u_p})$

$= \frac{3}{2} - \frac{1}{2} = \frac{2}{2} \quad p = 1 \lim p{\to}\infty\ \emptyset(u_p)$

$= \lim p{\to}\infty\ (u_p) = 1$ Clearly $\emptyset$ and ψ satisfy property (E.A.)

Example 11: Let $\mathbb{M}$ = [0, 1], Define F, ψ: $\mathbb{M} \to \mathbb{M}$ such that

$\emptyset(u) = 1 - u$ and $\psi(u) = u$ Consider a sequence

$up = \frac{1-1}{p}$ we have, $\lim p{\to}\infty\ \emptyset(u_p)$

$= \frac{1-1}{1-p} = 0$ and $\lim p{\to}\infty\ (u_p)$

$= \frac{1-1}{p} = 1 \lim p{\to}\infty\ \emptyset(u_p) \neq \lim p{\to}\infty\ \psi(u_p)$

Clearly $\emptyset$ and ψ do not satisfy property (E.A.)

Theorem 12.: Four self-mappings F, G, H and L be defined on Cone Banach Space ($\mathbb{M}$, ||·||) with the norm $\|u\| = d(u, 0)$ satisfying the conditions $\|Hu - Lv\| \leq a\|Fu - Hu\| + \|Fu - Lv\| + c\|Gv - Lv\|$ ... (1)

for all $u, v \in \mathbb{M}$ and $a, b, c \geq 0$, $a + 2b + c < 1$.

(i) $F(\mathbb{M}) \subseteq G(\mathbb{M})$ and $H(\mathbb{M}) \subseteq L(\mathbb{M})$

(ii) (F, H) and (G, L) are weakly compatible

(iii) One of pair (F, H) and (G, L) satisfies property (E.A.) Then F, G, H and L have a unique common fixed point

Proof: Let (G, L) satisfy property (E.A.) then there exist a sequence $\{up\}$ in (1) $\mathbb{M}$ such that

$\lim p{\to}\infty\ L\{u_p\} = \lim p{\to}\infty\ G\{u_p\}$

$= t$ for some $t \in \mathbb{M}$ Since $(\mathbb{M}) \subseteq (\mathbb{M})$ then ∃ a sequence $\{v_p\}$ in $\mathbb{M}$ such that

$L\ \{u_p\} = \{v_p\}$ Hence, $\lim p{\to}\infty\ \{vp\} = t$, We claim that

$\lim p{\to}\infty\ H\{v_p\} = t$

on the contradiction, we put $u = v_p$ and $v = u_p$ in (1)

we have $\|Hv_p - Lu_p\| \leq \|Fv_p - Hv_p\| + \|Fu_p - Lu_p\| + \|Gu_p - Lu_p\|$

From the above condition we get, ||

$Hv_p - Lu_p\| \leq a\|Fv_p - Hv_p\| + b \cdot 0 + c\|Gu_p - Lu_p\|$ Now taking

limit $p \to \infty$ $\|Hv_p - t\| \leq a\|t - Hv_p\| + c\|t - t\|$ $(1 - a)\|Hv_p - t\| \leq 0$

But $(1 - a) \neq 0$ then $\lim p{\to}\infty\ \{v_p\}$

$= t$ Hence $\lim p{\to}\infty\ \{v_p\}$

$= \lim p{\to}\infty\ \{v_p\}$

$= t$ Suppose first that, $(\mathbb{M})$ is complete subspace of $\mathbb{M}$ then

$t = (w)$ for some $w \in \mathbb{M}$ then $\lim p{\to}\infty\ \{u_p\}$

$= \lim p{\to}\infty\ \{v_p\}$

$= \lim p{\to}\infty\ \{u_p\}$

$= \lim p\to\infty \{v_p\} = t$

$= (w)$ We claim that,

$Hw = Fw$, on the contradiction we put $u = w$ and $v = u_p$ in (1) $\|$

$Hw - Lu_p\| \le \|Fw - Hw\| + \|Fw - Lu_p\| + \|Gu_p - Lu_p\|$

Letting $p \to \infty$ and using above condition $\|$

$Hw - t\| \le \|t - Hw\| + \|Fw - Fw\| + t - t\| \ \|$

$Hw - t\|(1 - a) \le 0$ But $(1 - a) \ne 0$ hence $Hw =$ t

Clearly, $Hw = Fw = t$.

Hence w is coincidence point of (H, F).

Now the weak compatibility implies

$HFw = FHw$ as $Ht = Ft$ on the other hand

$(\mathbb{M}) \subseteq (\mathbb{M}) \ \exists \ z \in \mathbb{M}$ such that $Hw = Gz$ thus

$Hw = Fw = Gz = t$

Let us show that z is coincidence point of (L, G) that is

$Gz = Lz = t$ if not then putting $u = w$ and $v = z$ in (1)

we have $\|Hw - Lz\| \le \|Fw - Hw\| + \|Fw - Lz\| + \|Gz - Lz\|$

Letting $p \to \infty$ and using above condition $\|$

$t - L \quad \| \le a \cdot 0 + b\|t - Lz\| + \|t - Lz\| \ (1 - b - c)\|t - Lz\| \le 0$ But $(1 - b - b) \ne 0$

hence $Lz = t$ Clearly, $Lz = Gz = t$.

Hence z is common coincidence point of (L, G).

Now the weak compatibility of pair (L, G). Implies that

$GLw = L \quad w = Gt = Lt$.

Therefore, t is common coincidence point of F, G, H and L. In order to show that t is common fixed point, let us show that t is common fixed point of F, G, H and L.

Let us substitute $u = w$ and $v = t$ in (1)

$\|Hw - Lt\| \le \|Fw - Hw\| + \|Fw - Lt\| + \|Gt - Lt\|$

From above conditions $\|t - Lt\| \le \ - t\| + \|t - Lt\| + \|Lt - Lt\| \ (1 - b)\|t - Lt\| \le 0 \ (1 - b) \ne 0$

then $Lt = t$ Clearly,

$Ft = Ht = Lt = Gt = t$.

Hence, t is common fixed point of F, G, H and L.

Similar, argument arises if we assume that $(\mathbb{M})$ is complete subspace $\mathbb{M}$.

Similarly the property (E.A.) for the pair (H, F) will give same results.

For uniqueness, we suppose that t' be another fixed point. Let us put

$u = t'$ & $v = t$ in (1),

then $\|Ht' - Lt\| \le \|Ft' - Ht'\| + \|Ft' - Lt\| + \|Gt - Lt\|$

$\|t' - t\| \le \|t' - t'\| + \|t' - t\| + \|t - t\| \ (1 - b)\|t' - t\| \le 0$

But $(1 - b) \ne 0$ Hence $t' = t$ Therefore, t is unique common fixed point of F, G, H and L.

Fisher B. in Math. Seminor Notes, Kobe University, 10 (1982) [23] obtains the E.A. property in the generalization of non-compatible maps, but it requires either the continuity of the map or the completeness of the entire space or any of the range spaces. A new concept of the CLR property (common limit range property) that does not impose such limits was recently put out by Japonica, Cho Y.J., [11].

Theorem 13. Let S and T be mappings into themselves that satisfy the requirement of a complete multiplicative metric space (X, d).

S(X) ⊂ X, T(X) ⊂ X

$d(s_x, t_y) \leq \{max.\left\{\frac{d(x,s_x)d(y,s_x)+d(y,t_y)]}{1+d(s_x,t_y)}\right\}.$

$, \frac{d(y,s_x)d(x,t_y)+d(x,y)d(s_x,y)}{d(s_x,t_y)+d(s_x,y)}\},$

$$\frac{d(x,s_x)d(y,s_x) + d(x,y)d(s_x,t_y)}{d(y,t_y) + d(y,s_x)},$$
$$\frac{d(y,s_x)d(x,t_y) + d(x,t_y)d(y,s_x)}{d(y,t_y) + d(y,s_x)}\}\}$$

for every x, where y ∈ X and λ ∈ (0, 1 2). In that case, S and T share a unique fixed point. We mostly expand and enhance theorem 4.1.2 in this section. By the way, we simplify the theorem equation (i) proof. First, we present a different straightforward proof of Theorem (i).

Theorem 14. Let S and T be mappings into themselves that satisfy the requirement of a complete multiplicative metric space (X, d).

S(X) ⊂ X, T(X) ⊂ X

$d(s_x, t_y) \leq \{max.\{\frac{d(x,s_x)d(y,s_x)+d(y,t_y)}{1+d(s_x,t_y)},$

$$\frac{d(y,s_x)d(x,t_y) + d(x,y)d(s_x,y)}{d(s_x,t_y) + d(s_x,y)},$$
$$\frac{d(x,s_x)d(y,s_x) + d(x,y)d(s_x,t_y)}{d(y,t_y) + d(y,s_x)},$$
$$\frac{d(y,t_y)d(x,t_y) + d(x,t_y)d(y,s_x)}{d(y,t_y) + d(y,s_x)}\}\}$$

for all x, y ∈ X, where λ ∈ (0, 1 2). Then S and T have unique common fixed point. Proof. Let x_0 be an arbitrary point in X, Since

S(X) ⊆ X and T(X) ⊆ X

we construct a sequence $\{x_n\}$ in X, such that

$x_{2n+1} = s_{x_{2n}}$ and x_{2n+2} = T x_{2n+1}∀n ≥ 0 (4.2.1.3) Suppose s_x = x, and t_y = y then from (4.2.1.2)

$d(s_x, t_y) = d(x, y) \leq \{max.\{\frac{d(x,y)[d(y,x)+d(y,y)]}{1+d(x,y)},$

$\frac{d(y,x)d(x,y)+d(x,y)d(x,y)}{d(x,y)+d(x,y)},$

$$\frac{d(x,x)d(y,x) + d(x,y)d(x,y)}{d(y,y) + d(y,x)},$$
$$\frac{d(y,y)d(x,y) + d(x,y)d(y,x)}{d(y,y) + d(y,x)}\}\}$$

= {max.{1,d(x , y),d(x , y),d(x , y)}}

d(x,y)≤ $\{d(x,y)\} < d(x,y) a contraduction\ if\ x \neq y$

There fore x=y

therefore s_x = T y. Suppose s_x = x put t_y = y in (i). Then

$d(s_x, t_y) = d(x, t_x) \leq \{max\{\frac{d(x,x)[d(x,x)+d(x,t_y)}{1+d(x,t_y)},$

$$\frac{d(x,x)d(x,t_x) + d(x,x)d(x,x)}{d(x,x)d(x,t_x) + d(x,x)},$$

$$\frac{d(x,x)d(x,x)+d(x,x)d(x,t_x)}{d(x,t_x)+d(x,x)},$$
$$\frac{d(x,t_x)d(x,t_x)+d(x,t_x)d(x,x)}{d(x,t_x)+d(x,x)}\}\}$$

= max{1,1,1,d(x,t_x)}}

d($s_x, t_y) = d(x, t_x) \leq \{d(x,t_x)\} < d(x,t_x) acotraduction , if\ x \neq t_x$

X = t_x te . y= t_y

As a result, S and T have identical sets of fixed points. Let x and y be S and T's shared fixed points.

Then s_x = x = t_x and t_y = y = t_y

⇒ s_x = t_y

∴ S and T have unique common fixed point. Put y = s_x in (4.2.1.2), Then

d($s_x, ts_x) \leq \{max.\{\frac{d(x,s_x)[d(s_x,s_x)+d(s_x,ts_x)]}{1+d(s_x,ts_x)}$,

∴ S and T have unique common fixed point. Put y = Sx in (4.2.1.2), Then

D($s_x, ts_x) \leq \{max.\{\frac{d(x,s_x)[d(s_x,s_x)+d(s_x,ts_x)]}{1+d(s_x,ts_x)}$,

$$\frac{1.\mathrm{d}(\mathrm{x},\mathrm{ts_x})+\mathrm{d}(\mathrm{x},\mathrm{s_x}).1}{\mathrm{d}(\mathrm{s_x},\mathrm{ts_x})+1},$$
$$\frac{\mathrm{d}(\mathrm{x},\mathrm{s_x}).1+\mathrm{d}(\mathrm{x},\mathrm{s_x})\mathrm{d}(\mathrm{x},\mathrm{s_x})\mathrm{d}(\mathrm{s_x},\mathrm{ts_x})}{\mathrm{d}(\mathrm{s_x},\mathrm{ts_x})+1},$$
$$\frac{\mathrm{d}(\mathrm{s_x},\mathrm{ts_x})\mathrm{d}(\mathrm{x},\mathrm{ts_x})+\mathrm{d}(\mathrm{x},\mathrm{ts_x}).1}{\mathrm{d}(\mathrm{s_x},\mathrm{ts_x})+1}\}\}$$

={max{d(x, s_x), $\frac{\mathrm{d}(\mathrm{x},\mathrm{s_x})+\mathrm{d}(\mathrm{x},\mathrm{s_x})}{1+\mathrm{d}(\mathrm{v},\mathrm{ts_x})}$,

$$\mathrm{d}(\mathrm{x},\mathrm{s_x}),\mathrm{d}(\mathrm{x},\mathrm{ts_x})\}\}$$

Now $\frac{\mathrm{d}(\mathrm{x},\mathrm{ts_x})+\mathrm{d}(\mathrm{x},\mathrm{s_x})}{1+\mathrm{d}(\mathrm{s_x},\mathrm{ts_x})} = \frac{\mathrm{d}(\mathrm{x},\mathrm{s_x})[\mathrm{d}(\mathrm{s_x},\mathrm{ts_x})+1]}{1+\mathrm{d}(\mathrm{s_x},\mathrm{ts_x})}$ = d(x, s_x)

From equation (i) we have $\mathrm{d}(\mathrm{s_x},\mathrm{ts_x}) \leq \{\max.\{\mathrm{d}(\mathrm{x},\mathrm{s_x}),\mathrm{d}(\mathrm{x},\mathrm{ts_x})\}\}$

$$\leq \{\max.\{\mathrm{d}(\mathrm{x},\mathrm{s_x}),\mathrm{d}(\mathrm{x},\mathrm{s_x}),\mathrm{d}(\mathrm{s_x},\mathrm{ts_x})\}\}$$

={$\mathrm{d}(\mathrm{x},\mathrm{s_x}),\mathrm{d}(\mathrm{s_x},\mathrm{ts_x})$}

Example 15. Consider the functions x(t) = t 2 and y(t) = t in [1, 2]. Then

$$\begin{aligned} d_I(x,y) &= \left[\min_{t\in[a,b]}|x(t)-y(t)|, \max_{t\in[a,b]}|x(t)-y(t)|\right] \\ &= \left[\min_{t\in[1,2]}\left|t^2-t\right|, \max_{t\in[1,2]}\left|t^2-t\right|\right] \\ &= [0,2]. \end{aligned}$$

Importance of Diameter Distance in Interval Metric :

This article discusses the importance of including diameter distance in interval metrics. Prior research on the interval metric by Lyra et al. [64], Santana et al. [92], and Trindade et al. [106] has mostly concentrated on determining the interval's greatest value in order to confirm its triangle inequality axiom and, consequently, the shortest path between locations. The influence of the interval's lower bound has no bearing on the fundamental criteria of the interval metric since the work is performed on a unique set of points. However, the two limits for functions are the highest and lowest of the distinction between the two roles. Since the difference between two functions is a function in and of itself, the minimum and maximum values will change when another function is

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positioned between two functions. The greatest difference will keep increasing when a third function is added. However, in the majority of situations, the minimal value usually falls. In other words, since the lower limit influences the distance between functions, merely taking into account the upper bound of the difference of functions would not yield the shortest path. This is where the diameter distance comes into play. Here is a verification of this case.

Example 16. :

Let $x(t) = t$ *and* $y(t) = 2t$ *in* $[a, b] = \left[\frac{1}{\sqrt{2}}, 1\right]$. *Take* $z(t) = \frac{1}{t}$.

$$\begin{aligned}\min_{t\in[a,b]} |x(t) - y(t)| &= \min_{t\in\left[\frac{1}{\sqrt{2}},1\right]} |t - 2t| \\ &= |-\frac{1}{\sqrt{2}}| \\ &= \frac{1}{\sqrt{2}}.\end{aligned}$$

$$\begin{aligned}\min_{t\in[a,b]} |x(t) - z(t)| &= \min_{t\in\left[\frac{1}{\sqrt{2}},1\right]} \left|t - \frac{1}{t}\right| \\ &= |1 - 1| \\ &= 0.\end{aligned}$$

$$\begin{aligned}\min_{t\in[a,b]} |z(t) - y(t)| &= \min_{t\in\left[\frac{1}{\sqrt{2}},1\right]} \left|\frac{1}{t} - 2t\right| \\ &= |\sqrt{2} - \sqrt{2}| \\ &= 0.\end{aligned}$$

That is,

$$\min_{t\in[a,b]} |x(t) - y(t)| \geq \min_{t\in[a,b]} |x(t) - z(t)| + \min_{t\in[a,b]} |z(t) - y(t)|.$$

By using the interval's diameter distance instead of the Moore interval module, the updated definition more closely investigates the metric properties. The length of the line segment connecting an interval's two extremes is known as its diameter. In this case, |dl(x, y)|diam represents the diameter distance of dl(x, y), which is defined as follows:

$$|d_l(x, y)|_{diam} = \max_{t\in[a,b]} |x(t) - y(t)| - \min_{t\in[a,b]} |x(t) - y(t)|.$$

The next definition is a modification of the definition of interval metric using the diameter distance defined above.

Definition 17. Let X = C[a, b] be the function space. The interval metric in C[a, b], we denote it by dl, is defined such that dl : X × X → IR as:

$$d_l(x, y) = \left[\min_{t\in[a,b]} |x(t) - y(t)|, \max_{t\in[a,b]} |x(t) - y(t)|\right],$$

for all functions $x, y \in C[a, b]$,

(i)$0 \in dl(x, x)$ (Reflexivity);

(ii) $dl(x, y) = dl(y, x)$ (Symmetry);

(iii) $|dl(x, y)|diam \leq |dl(x, z)|diam + |dl(z, y)|diam$ (Triangle inequality);

(iv) If $0 \in dl(x, y) = dl(x, x) = dl(y, y)$ then $x = y$ (Indiscernible identity).

Therefore, anytime we analyze the scenario of an interval metric with a diameter distance, the metric qualities can be visualized as just a line segment.

Remark 18. According to the triangle inequality property, the length of the line segment that connects the function x(t) and y(t)'s maximum and minimum will always be less than or equal to the total of the lengths of the two line segments that connect the functions (x(t), z(t)) and (z(t), y(t)).

II. CONCLUSION :

Stefan Banach's discovery of the Banach fixed point theorem in 1922 marked a significant turning point in the development of fixed point theory [14]. The existence and uniqueness of fixed points of specific self-mappings in a complete metric space under contractive mapping are guaranteed by this theorem. Numerous authors have so far produced intriguing extensions and generalizations of the Banach contraction principle based on its significance and simplicity. As seen in the proof, it also provides us with a constructive method for obtaining increasingly better approximations of the fixed point. This paper establishes the Banach fixed point theorem in C[a, b] using the interval metric with diameter distance.

In 1922, Stefan Banach developed the idea of Banach space and came up with a fixed point theorem for contraction mappings. The Banach fixed point theorem is sometimes referred to as the contraction mapping principle or the contraction mapping theorem. It is a crucial instrument in metric space theory. In addition to offering a constructive way to locate those fixed points, it ensures the existence and uniqueness of fixed points of specific self-maps of metric spaces. Numerous modifications of the Banach contraction principle have been made recently, reducing its hypothesis but maintaining the iterates' ability to converge to the mapping's unique fixed point. The Brouwer fixed point theorem is extended to topological vector spaces by the Schauder fixed point theorem, which may be of endless space. It states that if T is a continuous mapping of K into itself such that T(K) is contained in a compact subset of K and K is a convex subset of a topological vector space V, then T has a fixed point. The theorem was about fuzzy metric spaces in many ways. Fuzzy metric spaces were first proposed by Kramosil and Michalek [36] in terms of the t-norm. Schweitzer and Sklar [49] defined a probabilistic metric space using the notion of norm.

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