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APPLICATIONS OF LAPLACE TRANSFORMATION IN MECHANICAL ENGINEERING

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ABSTRACT

Laplace Transform is a very unique and powerful mathematical technique with its applications been applied in every field of engineering. This paper discusses with what Laplace Transform is, and what is it actually used for. The definition of Laplace Transform and most of its important properties have been mentioned. This paper also includes a brief overview of Inverse Laplace Transform. A number a methods used to find the time domain function from its frequency domain equivalent have been explained with detailed explanations. Also included the Applications of Laplace Transformation in Mechanical Engineering with key focus on the mathematical modelling of a Mechanical Vibratory System (Spring-Mass-Damper System) and its usage in Control System to obtain the transfer function and transient response of a mechanical system.

Keywords: Laplace Transform, Mechanical Engineering, Vibratory System, Control System.

I. INTRODUCTION

In mathematics, Laplace transformation constitutes an important tool in solving linear ordinary differential equations and partial differential equations in the time domain. These solutions are in respect with constant coefficients under suitable initial and boundary conditions with first finding the general solution and then evaluating from it the arbitrary constants.

As a significant tool the principal task of Laplace transform is, in establishing the suitable mathematical model for the solution of equations. Laplace transform converts the function f (t) from its time domain to frequency domain F(s). Then inverse Laplace transform transfers the converted frequency domain F(s) into time domain. In brief, Laplace transform converts differential or integral equations into an algebraic equation. The extensive choice of application makes Laplace transform as a powerful tool in studying the characteristics of engineering problems. [2].

Mathematically, it can be expressed as:

$$\mathbf{L_t[f(t)]} = \int_{\mathbf{0}}^{\infty} \mathbf{e^{-st}f(t)dt} = \mathbf{F(s)}$$

Laplace transform methods have a key role to play in the modern approach to the analysis and design of engineering systems [5]. This paper helps the reader understands the fundamentals of Laplace transformation and presents basic concept of the applications of Laplace Transformation in mechanical engineering with respect to mathematical modeling of mechanical systems.

II. HISTORY

Laplace transformation, in mathematics, a particular integral transformation invented by the French mathematician Pierre-Simon Laplace (1749-1827), and systematically developed by the British physicist Oliver Heaviside (1850-1925), to simplify the solution of many differential equations that describe physical processes. The current widespread use of the transform (mainly in engineering) came about during and soon after World

War II, replacing the earlier Heaviside operational calculus. The advantages of the Laplace transform had been emphasized by Gustav Doetsch, to whom the name Laplace Transform is apparently due. From 1744, Leonhard Euler investigated integrals of the form

$$Z = \int X(x).e^{ax}dx \text{ and } Z = \int X(x).e^{-a}dx$$

as solutions of differential equations but did not pursue the matter very far. Joseph Louis Lagrange was an admirer of Euler and, in his work on integrating probability density functions, investigated expressions of the form

$$\int X(x)e^{-ax}a^{x}dx$$

which some modern historians have interpreted within modern Laplace transform theory. These types of integrals seem first to have attracted Laplace's attention in 1782, where he was following in the spirit of Euler in using the integrals themselves as solutions of equations. However, in 1785, Laplace took the critical step forward when, rather than simply looking for a solution in the form of an integral, he started to apply the transforms in the sense that was later to become popular. He used an integral of the form

$$\int X^{s}\emptyset(s)dx$$

in to a Mellin transform, to transform the whole of a difference equation, in order to look for solutions of the transformed equation. He then went on to apply the Laplace transform in the same way and started to derive some of its properties, beginning to appreciate its potential power. Laplace also recognized that Joseph Fourier's method of Fourier series for solving the diffusion equation could only apply to a limited region of space, because those solutions were periodic. In 1809, Laplace applied his transform to find solutions that diffused indefinitely in space [4].

III. PRELIMINARIES

In order for any function of time f (t) to be Laplace transformable, it must satisfy the following Dirichlet conditions:

- f (t) must be piecewise continuous which means that it must be single-valued but can have a finite number of finite isolated discontinuities for t > 0.
- f (t) must be an exponential order which means that f(t) must remain less than Se^{-ta_o} as t approaches ∞ where S is a positive constant and a_o is a real positive number.

If there is any function f (t) that satisfies the Dirichlet conditions, then

$$F(s) = \int_{0}^{\infty} f(t)e^{-st}dt$$

Written as $L(f(t))$ is called the Laplace transformation of f (t). Here, 's' can be either a real variable or a complex quantity [1].

The integral $\int f(t)e^{-st}dt$ converges if

$$\int |f(t)e^{-st}|dt < \infty, s = \sigma + jw \text{ [1].}$$

Various Properties of Laplace Transformations

1. Linearity Property
2. Shifting Property
3. Change of Scale Property
4. Multiplication by t^n Property
5. Laplace Transform For Derivative
6. Laplace Transform For Integral
7. Initial Value Theorem
8. Final Value Theorem

Linearity Property: If f (t) and g (t) are any two functions of t whose Laplace transforms exist, and a, b are any two constant coefficients, then we have

$$L [a f (t) + b g (t) = a L [f (t) + b L [g (t)]$$

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Example: $L\{sinh(t)\} = L\left\{\frac{1}{2}e^{t} - \frac{1}{2}e^{-t}\right\} = \frac{1}{2}L\{e^{t}\} - \frac{1}{2}L\{e^{-t}\} = \frac{1}{2}\left\{\frac{1}{(s-1)} - \frac{1}{(s+1)}\right\}$

$= \frac{1}{2}\{\frac{(s+1)-(s-1)}{(s^2-1)} = \frac{1}{(s^2-1)}$ [6].

Shifting Property: If $L[f(t)] = F(s), then\ L[e^{at} f(t)] = F(s-a)$

Thus, a in the above formulation is the shifting factor, i.e. the parameter s in the transformed function f (t) has been shifted to (s-a).

Example: Perform the Laplace transform on function: $F(t) = e^{3t}Sin(at)$

$$L[f(t)] = L[Sin\ at] = \frac{a}{s^2 + a^2}$$

We use the shifting property to get the Laplace transform of $F(t) = e^{3t}Sin(at)$, by shifting the parameter s by 3,

$$L[F(t)] = L[e^{3t}\ Sin\ at] = \frac{a}{(s-3)^2 + a^2}$$ [11].

Change of Scale Property: If L [f (t)] = F(s), then $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

where a = scale factor for the change

Example: Perform the Laplace transform of function F (t) = Sin3t

$$L[f(t)] = L[Sint] = \frac{1}{s^2+1} = F(s)$$

We may find the Laplace transform of F (t) using the Change scale property to be:

$$L[Sin2t] = \frac{1}{3}\frac{1}{\left(\frac{s}{3}\right)^2+1} = \frac{3}{s^2+9}$$ [11]

Multiplication by tⁿ Property: If $L[f(t)] = F(s), then$

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[F(s)]$$ [5].

Laplace Transform For Derivative: For t≥0, let f (t) be function which is having an exponential order 'a' and is continuous on [0, ∞), [so that lim e^{-st}f (t) = 0]. Let f' (t) be also continuous and of exponential order 'a' or piecewise continuous on [0, ∞), then

$$L\,f'(t) = sF(s) - f(0) for\ s > a, where\ Lf(t) = F(s)$$

Initially we have derivative of time domain then we convert it into algebraic equation in Laplace domain [6].

Laplace Transform For Integral: The integration theorem includes:

$$If\ L[f(t)] = F(s), then\ L\int_0^t f(u)du = \frac{1}{s}F(s)$$

Evaluation of integral:

$$If\ L[f(t)] = F(s)\ i.e., \quad \int_0^\infty e^{-st} f(t)dt = F(s)$$

Taking the limits as s→0

$$\int_0^\infty f(t)dt = F(0)$$

By assuming the integral to be convergent... [6]

Initial Value Theorem:

$$If\ L[f(t)] = F(s), then$$

$$\lim_{t\to 0}[f(t)] = \lim_{s\to\infty} sF(s)$$

Conditions: 1. Applicable only when f (t) = 0, t<0

2. Applicable only when we have the power of numerator polynomial less than that of denominator i.e.

$$F(s) = \frac{1}{s+1}$$ [6].

Final Value Theorem:

This theorem is not quite as useful as the initial value theorem, for it can be used only with a certain class of transforms. In order to determine whether a transform fits into this class, the denominator of F(s) must be evaluated to find all values of s for which it is zero.

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$$If\ L[f(t)] = F(s)\ then,$$

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Increasing exponentials (like $e^{at}$where 'a'is a positive number) that goes to ∞ as t increases, and the oscillating functions (like sine and cosine that have no final value) are examples for which this theorem can't be used [6].

Inverse Laplace Transformation

The inverse Laplace transformation is the transformation of a Laplace transformation into a function of time. If$L[f(t)] = F(s)$, thenf (t) is the inverse Laplace transformation of F(s), the inverse being written as: $f(t) = L^{-1}[F(s)]$. Here L^{-1} denotes the inverse Laplace transformation.

Example: Since $L[e^{2t}] = \frac{1}{s-2}, then\ we\ have\ L^{-1}\left[\frac{1}{s-2}\right] = e^{2t}$ [6]

3 Ways to inverse Laplace transform:

- Use LP Table by looking at F(s) in right column for corresponding f (t) in middle column-chance of success is not very good.
- Use partial fraction method for F(s) = rational function (i.e. fraction functions involving polynomials), and
- The convolution theorem involving integrations [11].

IV. APPLICATIONS IN MECHANICAL ENGINEERING

Laplace Transform plays a very huge and important role in the field of Mechanical Engineering. It ranges from its application to mechanical vibratory system to the mathematical modeling of mechanical systems and as well as finding transfer function of a control system.

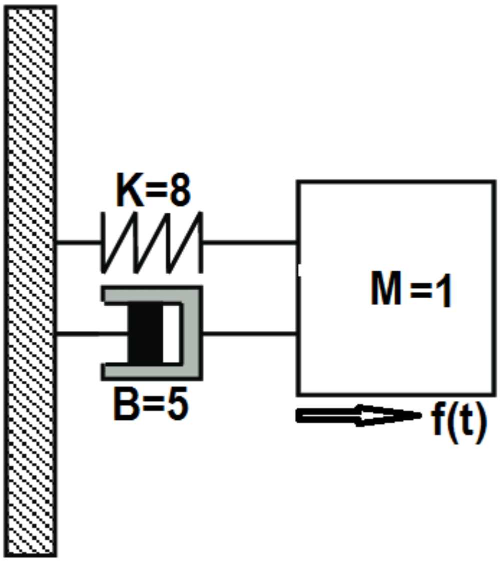
A. Mechanical Vibratory System:

In Mechanical Engineering, Vibratory systems include three basic components: Spring (K), Mass (M) and Damper (B). The spring works as a means of storing potential energy, and the mass is for storing kinetic energy. When the spring receives the force through the application of the mass, energy is lost gradually or absorbed by the damper in the system. In the Spring-Mass-Damper System, there is either a transitional displacement or a rotational displacement.

Translational Displacement: a translational displacement of an object is when its movement from one point to another is along a linear path.

VIBRATORY SYSTEM COMPONENTS: TRANSLATIONAL DISPLACEMENT		
COMPONENT	**VELOVITY [v(t)]**	**DISPLACEMENT [x(t)]**
SPRING (K)	$f(t) = k\int_0^t v(t)dt$	$f(t) = k\ x(t)$
MASS (M)	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$
DAMPER (B)	$f(t) = \mathrm{B\ v(t)}$	$f(t) = B\frac{dx(t)}{dt}$

Example 1: Consider the below Spring-Mass-Damper System with transitional displacement:



Here the system can be modeled by solving the differential equation of the system using the application of Laplace Transform. Herein, M(Mass of block) = 1, K(Stiffness of Spring) = 8 and B (Damper) = 5.

$$f(t) = M\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + K\,x(t)$$

Taking Laplace Transform:

Initially when $t = 0, \frac{dx}{dt} = 0, x(t) = 0$

$$F(s) = M\,S^2\,X(s) + B\,S\,X(s) + K\,X(s)$$

Sustituting the values for M, B and K

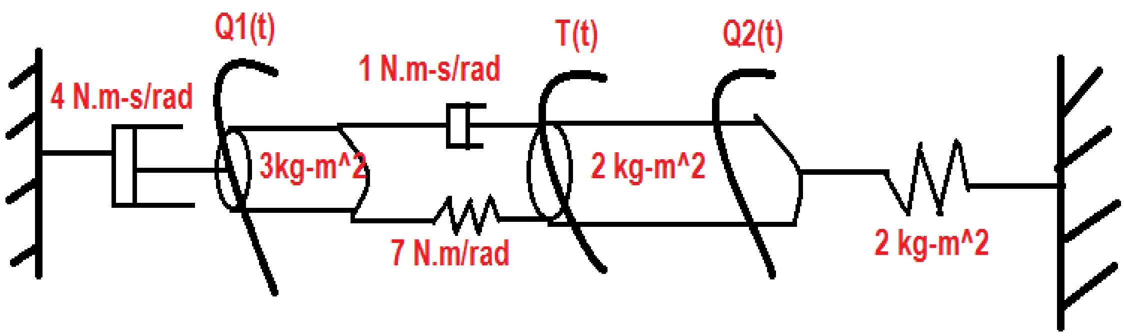
$$F(s) = S^2\,X(s) + 5\,S\,X(s) + 8\,X(s)$$

Rotational Displacement: an object is said to be rotationally displaced when its movement is around the axis of the object in a circular path.

VIBRATORY SYSTEM COMPONENTS: ROTATIONAL DISPLACEMENT		
COMPONENT	**VELOVITY [w(t)]**	**DISPLACEMENT [Θ(t)]**
SPRING (K)	$f(t) = k\int_0^t w(t)dt$	$f(t) = k\,\Theta(t)$
INERTIA (J)	$f(t) = J\frac{dw(t)}{dt}$	$f(t) = J\frac{d^2\Theta(t)}{dt^2}$
DAMPER (D)	$f(t) = \mathrm{D\,w(t)}$	$f(t) = D\frac{d\Theta(t)}{dt}$

Example 2: Consider the Spring-Mass-Damper System with Rotational Displacement

Using Laplace Transform to find the Transfer Function of a Rotational Mechanical System below:



This system shows a rotational displacement system with two equations of motion. The system at a fix point contains three springs and two dampers. Here a transfer function is found uisng the technique of Laplace Transform.

$$\boldsymbol{G(s)} = \frac{\Theta_1(\boldsymbol{t})}{\boldsymbol{T(s)}}$$

Considering the two equations of motion Θ_1 and Θ_2 while taking Laplace Transform, we get:

$$\Theta_1 \rightarrow (3s^2 + 4s + 1s + 7)\Theta_1(s) - (1s + 7)\Theta_2 = 0$$

$$\Theta_2 \rightarrow -(1s + 7)\Theta_1(s) + (2s^2 + 1s + 7 + 2)\Theta_2(s) = T(s)$$

$$(3s^2 + 5s + 7)\Theta_1(s) - (s + 7)\Theta_2(s) = 0$$

$$-(s + 7)\Theta_1(s) + (2s^2 + s + 9)\Theta_2(s) = T(s)$$

$$\Theta_1(s) = \frac{\begin{matrix} 0 & -(s+7) \\ T(s) & (2s^2 + s + 9) \end{matrix}}{\begin{matrix} 3s^2 + 5s + 7 & -(s+7) \\ -(s+7) & 2s^2 + s + 9 \end{matrix}}$$

$$\Theta_1(s) = \frac{0 - \big(T(s)\big) - (s + 7)}{(3s^2 + 5s + 7)(2s^2 + s + 9) - (s + 7)^2}$$

$$= \frac{T(s)(s + 7)}{6s^4 + 3s^3 + 27s^2 + 10s^3 + 5s^2 + 45s + 14s^2 + 7s + 63 - s^2 - 14s - 49}$$

$$= \frac{T(s)(s + 7)}{6s^4 + 13s^3 + 45s^2 + 38s + 14}$$

$$G(s) = \frac{\Theta_1(s)}{T(s)} = \frac{(s + 7)}{6s^4 + 13s^3 + 45s^2 + 38s + 14}$$

B. Control Systems

A Control System is an integral part of Mechanical Engineering in modern society and Laplace Transform enhance the working of the subsystems assembled for the cause of acquiring a preferred output with preferred performance of a unique system input. With Control System in Mechanical Engineering, we can move and operate large equipment with precision that would in any other case be impossible. Robots are currently been used in many aspects of life to assist humans perform huge and difficult tasks. As supported by this current generation of technology, large assemblies in Mechanical Engineering are perform by robots, controlling those robots, mathematical modeling is needed and Laplace Transform is key as it is an integral method that convert the function, f (t) in the time domain to the function, F(s).

An application of Laplace Transform in control system could simple take in consideration the below differential equation in obtaining an expression for y (t).

$$\frac{d^2y(t)}{dt^2} + \frac{9dy(t)}{dt} + 20y(t) = 48e^{-t}$$

Taking the operation of Laplace Transform on both sides: Given that: $\frac{\mathrm{dy}}{dt} = y(t) = 0$

$$s^2y(s) + 9sy(s) + 20y(s) = \frac{48}{s + 1}$$

$$(s^2 + 9s + 20)y(s) = \frac{48}{s + 1}$$

$$y(s) = \frac{48}{(s + 1)(s^2 + 9s + 20)}$$

Consider the factor for $s^2 + 9s + 20 \ as \ (s + 5)(s + 4)$ and substitute in the above equation.

$$y(s) = \frac{48}{(s + 1)(S + 4)(s + 5)}$$

For each linear factor above consider one constant: A, B and C

$$y(s) = \frac{A}{s + 1} + \frac{B}{s + 4} + \frac{C}{s + 5}$$

$$\frac{48}{(s+1)(s+4)(s+5)} = \frac{A}{s+1} + \frac{B}{s+4} + \frac{C}{s+5}$$

Multiply by: $(s+1)(s+4)(s+5)$ to find A, B and C

$$48 = A(s+4)(s+5) = B(s+1)(s+5) + C(s+1)(s+4)$$

After simplifying using partial fraction method we get the following values

$$A = 4, B = -16 \; and \; C = 12$$

$$y(s) = \frac{4}{s+1} - \frac{16}{s+4} + \frac{12}{s+5}$$

Taking the Inverse Laplace Transform of y(s), we get:

$$y(t) = 4e^{-t} - 16e^{-4t} + 12e^{-5t}$$

V. CONCLUSION

As supported by this paper it is understood that the applications of Laplace Transform in Mechanical Engineering is very useful in solving complex mathematical modeling problems and provides the solution of different real life mechanical engineering problems. In this generation of technological advancement, large mechanical equipment are been controlled by computer's commands and human efforts are been replaced by robots, therefore, Laplace Transform is very essential in these days advancement, it supports effective operations of these mechanical system and provides easy solutions to their problems in Mechanical Engineering.

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VI. REFERENCES

[1] Manju Devi, "Laplace Transforms and its Applications", International Journal of Advance Research, Ideas and Innovations in Technology (IJARIIT), ISSN: 2454-132x (Volume 3, Issue 3 2017), pp 1213-1218.

[2] Vaithyasubramanian S*, K. Vinil Kumar, K. JosehpPranadeer Reddy; "Study on Applications of Laplace Transformation: A Review Engineering and Technology, 2018 |IIDABJ| Vol. 9 | S1 | pp 1-6.

[3] Ananda and Gangadharaiah Y. H, "Applications of Laplcce Transforms in Engineering and Economics". International Journal of Trend in Research and Development. Jan-Feb. 2016, volume 3(1), ISSN: 2394-9333, pp 358-361.

[4] Rahul M. Jetwani*, Ujivala Y. Gawarguru, Rajshree A. Naphade, Mitali K. Tibdewal, "Applications of Laplace Transformation in Engineering Field". 2nd National Conference Recent Innocations in Science and Engineering (NC-RISE 17), Volume: 5 Issue: 9 ISSN: 2321-8169. Pp 39-41.

[5] Prof. L. S. Sawant, "Applications of Laplace Transform in Engineering Fields", International Research Journal of Engineering and Technology (IRJET) Volume: 05 Issue: 05 | May-2018 pp 3100-3105.

[6] GaganGaba, "A Study on the Applications of Laplace Transformation". Journal of University of Shanghai for Science and Technology. Volume 23, Issue 8, August -2021. ISSN: 1007-6735 pp 84-91.

[7] Malkeet Singh Bhuller, "Study on Properties and Applications of Laplace Transformation: A Review. Pramana Research Journal. Volume 8, Issue 4, 2018. ISSN No: 2249-2976 pp 246-251.

[8] P. C. Jadhav, S. S. Sawant, O. S. Kunjir, T. A. Karanjkar, "On Noteworthy Applications of Laplace Transform in Real Life". International Journal of Innovative Science and Research Technology. Volume 5, Issue 3, March-2020. ISSN No: 2456-2165 pp 41-47.

[9] Alfred Daci, SaimirTola, "Laplace Transform, Application in Population Growth", International Journal of Recent Technology and Engineering (IJRTE) ISSN: 2277-3878, Volume-8 Issue-254, July 2019 pp 954-957.

[10] Dr. Ajaykumar J. Patel, "Study on Laplace Transformation and its Application in Science and Engineering Field". Internatioanal Jouranl of Creative Research Thoughts (IJCRT). Volume 8, Issue 5 May 2020 | ISSN: 2320-28820 pp 1985-1991.

[11] Tai-Ran Hsu, "ME 130 Applied Engineering Analysis: Review of Laplace Transform and Its Application in Mechanical Engineering Analysis (Condensed Version)," 2018 Version. Chapter 5: pp 1-39.

[12] Dr. Vishal Saxena, "Study of Various Applications of Laplace Transform." International Journal for Research Multidisciplinary Field, ISSN: 2455-0620 Volume-6, Issue-6, June-2020. Pp. 120-132.

[13] K. Joseph Pranadeer Reddy, Vaithyasubramnian S. "A Survey of Laplace Transform Applications in Various Fields of Science and Engineering." International Journal of Pure and Applied Mathematics. Volume 119, No. 7 2018, 769-777.

[14] Rudolf A. Treumann, and Wolfgang Baumjohann, "Fractional Laplace Transforms – A perspective." Perspective Article. June 2014 | Volume 2 | Article 29. Pp 1-4.

[15] Tai-Ran Hsu, "Introduction to the Laplace Transform and Its Applications." Applied Engineering Analysis. Chapter 6: 2018 (ISBN: 9781119071204) pp 1-44.

[16] Rinku Gouda, Tapas Kumar Mohanta, Biplab Kumar Rath, "Study on Laplace Transform." International Journal of Scientific Development and Research, ISSN: 2455-2631 June 2021 IJSDR | Volume 6 | Issue 6 | pp 200-205.

[17] J. L. E. da Silva, "New Perspectives of the k-Laplace Transform in Partial Differential Equations (PDE's). Lab. Of Quantum Information Technology, Department of Teleinformatic Engineering – Federal University of Ceara-DETI/UFC. May 2021.

[18] Karan Asher, "An Introduction to Laplace Transform." International Journal of Science and Research (IJSR), India online ISSN: 2319-7064 Volume 2 Issue 1, January 2013. Pp 601-606.

[19] P. VenkataPavani and U. Lakshmi Priya and B. Amarnath Reddy, "Solving Differential Equations by using Laplace Transforms." International Journal of Research and Analytical Reviews. [Volumee 5 | Issue 3 | July-Sept. 2018]. Pp 1796-1799.

[20] SarinaAdhikari, "Laplace Transforms and Its Applications." Department of Electrical Engineering and Computer Science, University of Tennessee.