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Is $\Gamma_{ve}(G) + \beta_0(G) \leq n$?

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ABSTRACT

In this paper, we study vertex-edge domination in graphs, and its parameters, where a vertex dominates the edges incident to it as well as the edges adjacent to these incident edges. We discuss some of the theorems provided in the 2015 paper on ve-domination by Razika et al., and some from the 2010 Ph.D. thesis by Lewis. We then answer in affirmative one of the five open questions posed by Boutrig et al. at the end of their 2015 paper.

Keywords: Vertex-Edge Domination, Dominating Sets, Bipartite Graphs, Graph Theory, Mathematics.

I. INTRODUCTION

Dominating sets is a relatively new topic in Graph theory, as mathematicians started studying it in 1950 onwards. The rate of research on domination significantly increased in the mid-1970s. In 1972, Richard Karp proved the set cover problem to be NP-complete. This had immediate implications for the dominating set problem, as there are straightforward vertex to set and edge to non-disjoint intersection bijections between the two problems. This proved the dominating set problem to be NP-complete as well. Dominating sets are of practical interest in several areas. In wireless networking, dominating sets are used to find efficient routes within ad-hoc mobile networks. They have also been used in document summarization, and in designing secure systems for electrical grids. In this paper, study a special kind of domination: Vertex-Edge Domination.

II. INTRODUCTION: TERMINOLOGY

We now provide the definitions needed for the rest of the paper. The trivial graph is the graph on one vertex. This graph meets the definition of connected vacuously (since an edge requires two vertices). A non-trivial connected graph is any connected graph that isn't this graph. Let $G = (V, E)$ be a graph with order $n = |V|$. The open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood is $N[v] = N(v) \cup v$. For a set $S \subseteq V$, the open neighborhood is $N(S) = \cup_{v \in S} N(v)$, the closed neighborhood is $N[S] = N(S) \cup S$, and $G[S]$ is the subgraph induced by the vertices of S. The private neighborhood of a vertex u with respect to S is defined as $pn[u, S] = \{v \in V \mid N[v] \cap S = \{u\}\}$. A vertex $u \in V$ is said to ve − dominate an edge $vw \in E$ if

1. $u = v$ or $u = w$, that is, u is incident to vw, or

2. uv or uw is an edge in G, that is, u is incident to an edge that is adjacent to vw.

A set $S \subseteq V$ is a vertex − edge dominating set (or simply a ve-dominating set) if for every edge $e \in E$, there exists a vertex $v \in S$ such that v ve-dominates e. We denote by $\gamma_{ve}(G)$ the minimum cardinality of a ve-dominating set of G called the vertex−edge domination number and by $\Gamma_{ve}(G)$ the maximum cardinality of a minimal ve-dominating set of a graph G called the upper vertex − edge domination number (or simply the upper 3 ve-domination number). A set $S \subset V$ is independent if no two vertices in S are adjacent. A set $S \subseteq V$ is an independent vertex-edge dominating set (or simply an independent ve-dominating set) if S is both independent and ve-dominating. Finally, independent vertex-edge domination number, $i_{ve}(G)$, of G is the minimum cardinality of an independent ve-dominating set and the upper independent vertex − edge domination number $\beta_{ve}(G)$ is the maximum cardinality of a minimal independent ve-dominating set of G.

III. INTRODUCTION: THEOREMS AND WORK DONE PREVIOUSLY ON RELATED PROBLEM

The authors of the paper Vertex-edge domination in graphs [2] went through the following propositions and theorems 4 introduced by lewis et al. in 2010[1]:

Proposition 1. [2] Let S be a ve-dominating set of an ntc graph G. Then S is a minimal ve-dominating set if and only if every vertex $v \in S$ has at least one private edge with respect to S.

Proposition 2. (Lewis et al.) Every vertex in a ve-irredundant set S of an ntc graph G has a private neighbor in V − S.

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Corollary 3. [1] Every vertex in a minimal ve-dominating set S of an ntc graph G has a private neighbor in V − S.

Theorem 4. (Lewis et al. [1]) For any ntc graph G of order n,

$ir_{ve}(G) \leq \gamma_{ve}(G) \leq i_{ve}(G) \leq \beta_{ve}(G) \leq \Gamma_{ve}(G) \leq IR_{ve}(G) \leq n/2$

Then they set upon answering these four questions raised by Lewis et al. in the 2010 paper:

For an ntc graph of order n,

1. Is $IR_{ve}(G) + \gamma(G) \leq n$?
2. Is $\Gamma_{ve}(G) + i(G) \leq n$?
3. Is $IR(G) + \gamma_{ve}(G) \leq n$?
4. Is $\Gamma(G) + i_{ve}(G) \leq n$?

Of these four questions, the most important to me was number 2 because the problem that I prove is related to it. The way they answer (2) is the following:

First they prove that for an ntc graph of order n, $IR(G)+ i_{ve}(G) \leq n$ if G is a star. Combining this result with the fact that $\gamma_{ve}(G) \leq i_{ve}(G)$ and $\Gamma(G) \leq IR(G)$ answer questions 3 and 4, and obtain the following corollary:

Corollary: If G is an ntc graph of order n, then $IR(G)+ \gamma_{ve}(G) \leq n$ and $\Gamma(G) + i_{ve}(G) \leq n$ if G is a star. Then they follow this up by proving that $IR_{ve}(G) + i(G) \leq n$ for any ntc graph G of order n.

Their proof this by forming two sets whose cardinalities are the ones that they're investigating. Then they use previously known facts and combine them to form inequalities which result in the parameters in question to add up to n, as desired. At the end of their paper they state the following:

"4. In Theorem 7, we prove that if G is an ntc graph of order n, then $IR_{ve}(G) + i(G) \leq n$. This raises the following question: **Is $\Gamma_{ve}(G) + \beta_0(G) \leq n$?**"

IV. PROOF

Theorem. If G is an ntc graph of order n then, $\Gamma_{ve}(G) + \beta_0(G) \leq n$.

Proof. We now prove the main result. Let A be a minimal ve-dominating set of G of maximum cardinality $\Gamma_{ve}(G)$. By proposition 2, each vertex of A has at least one private neighbor in V −A. Let D be an independent set of G of maximum cardinality β_0.

We consider three cases:

Case 1: D ⊆ A:

Then by Lewis et al. we have:

$ir_{ve}(G) \leq \gamma_{ve}(G) \leq i_{ve}(G) \leq \beta_{ve}(G) \leq \Gamma_{ve}(G) \leq IR_{ve}(G) \leq n/2$

$\therefore |A| \leq n/2 \Longrightarrow |D| \leq n/2$ (since $D \subseteq A$)

Hence, $|A| + |D| \leq |V| = n$.

Case 2: D ⊆ V − A so $A \cap D = \varphi$:

The two sets are disjoint, and $|A| \leq n/2$

Hence, $|A| + |D| \leq |V| = n$.

Case 3: Now we arrive at the most important case:

$A \cap D \neq \varphi$ AND $V - (A \cup D) \neq \varphi$

This means that $\forall_v \in A \cap D$ does not have a private neighbor in D. And that $\forall_v \in A \cap D$ does not have a private neighbor in A. Hence by corollary 3, all the private neighbors exist in $V - (A \cup D)$.

$\Rightarrow |A \cap D| \leq |V - (A \cup D)|$

$\therefore |A|+|D| = |A \cup D|+|A \cap D| \leq |A \cup D|+|V - (A \cup D)| =$

$|V| \Rightarrow \Gamma_{ve}(G) + \beta_0(G) \leq n$

This completes the proof.

V. CONCLUSION

We tried to find a family of graphs for which the equality $\Gamma_{ve}(G) + \beta_0(G) = n$ would hold, and we hypothesized that it would be true for all trees, however, in the construction of a tree from the equality itself was not achieved. It was also hypothesized that the equality holds for all bipartite graphs, but when we tried to construct a bipartite graph using the equality itself, we faced the problem of always having an odd cycle in our construction. So maybe the families of graphs are not just limited to trees and bipartite graphs, and future work could include working on and finding for what families of graphs the equality holds.

VI. REFERENCES

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