e-ISSN: 2582-5208
International Research Journal of Modernization in Engineering Technology and Science
Volume:02/Issue:05/May-2020 www.irjmets.com

QUANTUM LOGIC GATES: A DETAILED THEORETICAL STUDY OF THE HADAMARD GATE

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ABSTRACT

Quantum computers are meant to perform quantum computations. In order to understand how quantum computers perform quantum computations, we need to understand qubits' quantum states and how we can change those states. Quantum logic gates help do just that – change qubits' quantum states so that quantum computers can perform the desired quantum computations. This review paper is aimed at performing a detailed theoretical study of the HADAMARD Gate. The HADAMARD Gate is a quantum gate that very clearly demonstrates the involvement of quantum effects.

KEYWORDS: Quantum Computing, Quantum Computer, Qubit, Quantum Bit, Quantum Logic Gate, HADAMARD Gate.

I. INTRODUCTION

Quantum logic gates serve as the basic building blocks for quantum computers to perform quantum computations. Quantum logic gates manipulate quantum information by changing the quantum state(s) of a qubit or a collection of them. Just as classical computers employ the use of classical logic gates such as the AND, OR, NOT gates and the likes in order to perform different types of computations, quantum computers use quantum logic gates in order to perform quantum information processing tasks and quantum computations.

II. ANALYSIS

a) HADAMARD Gate: How does it act on a qubit's general quantum superposition state?

The HADAMARD Gate is generally denoted by **H**. We start off by showing how the HADAMARD Gate acts on the computational basis quantum states, $|0\rangle$ and $|1\rangle$.

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

However, in quantum computing, the computational basis quantum states, $|0\rangle$ and $|1\rangle$, are not the only quantum states; a qubit can be in more general quantum superposition states.

Let us consider that a qubit is in a general quantum superposition state, $\alpha\,|0\rangle + \beta\,|1\rangle$. The HADAMARD Gate acts linearly on the qubit's general quantum superposition state, $\alpha\,|0\rangle + \beta\,|1\rangle$.

Thus,

$$H(\alpha|0\rangle + \beta|1\rangle)$$

$= \alpha\, H\,|0\rangle + \beta\, H\,|1\rangle$

$$= \alpha\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) + \beta\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

Breaking the equation further down, we arrive at the following equation.

$$= \alpha\left(\frac{|0\rangle}{\sqrt{2}} + \frac{|1\rangle}{\sqrt{2}}\right) + \beta\left(\frac{|0\rangle}{\sqrt{2}} - \frac{|1\rangle}{\sqrt{2}}\right)$$

$$= \alpha\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] + \beta\left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right]$$

$$= \frac{\alpha}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{\beta}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$= \frac{\alpha}{\sqrt{2}}|0\rangle + \frac{\alpha}{\sqrt{2}}|1\rangle + \frac{\beta}{\sqrt{2}}|0\rangle - \frac{\beta}{\sqrt{2}}|1\rangle$$

$$= \left(\frac{\alpha}{\sqrt{2}}|0\rangle + \frac{\beta}{\sqrt{2}}|0\rangle\right) + \left(\frac{\alpha}{\sqrt{2}}|1\rangle - \frac{\beta}{\sqrt{2}}|1\rangle\right)$$

$$= \left[\left(\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{2}}\right)|0\rangle\right] + \left[\left(\frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{2}}\right)|1\rangle\right]$$

$$= \frac{\alpha + \beta}{\sqrt{2}}|0\rangle + \frac{\alpha - \beta}{\sqrt{2}}|1\rangle$$

Fig-1: Algebraic equation showing how the HADAMARD Gate acts on a qubit's general quantum superposition state

b) HADAMARD Gate: Quantum Circuit Representation

The Quantum Circuit representation of the HADAMARD gate looks like the following.

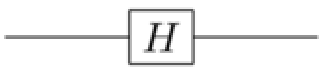


Fig-2: Quantum Circuit representation of the HADAMARD Gate

c) HADAMARD Gate: Matrix Representation

The matrix representation of the HADAMARD gate is as follows.

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Fig-3: Matrix Representation of the HADAMARD Gate

Let us now prove that the matrix representation of the HADAMARD gate is correct.

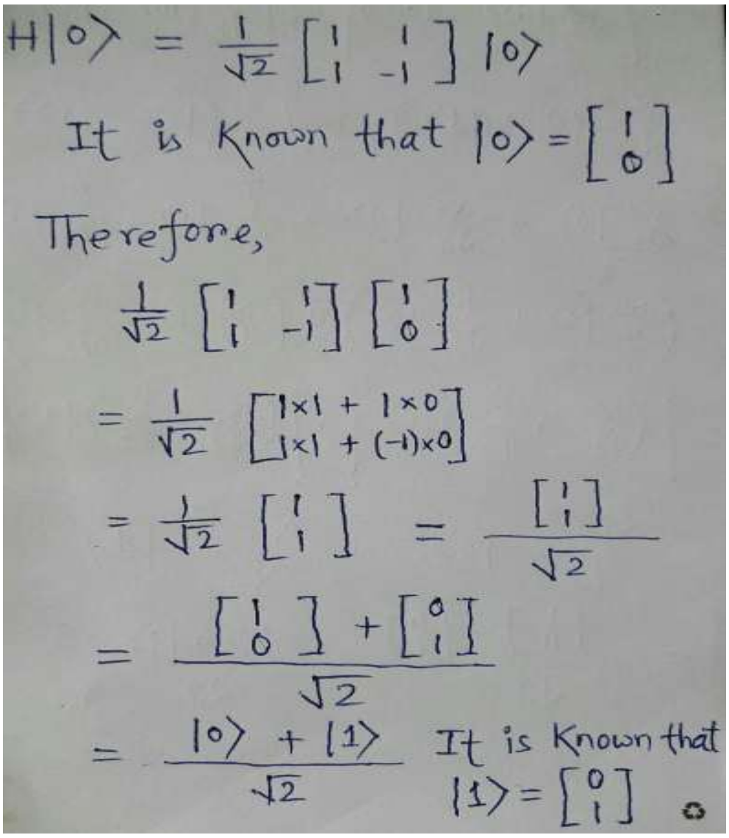


Fig-4: Mathematically proving that the matrix representation of the HADAMARD gate is correct

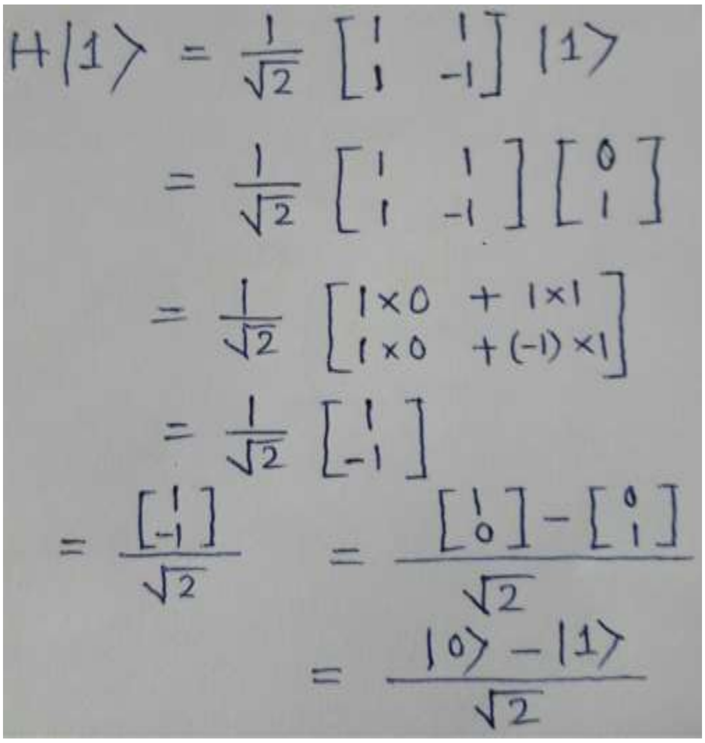


Fig-5: Mathematically proving that the matrix representation of the HADAMARD gate is correct

d) Analysis of a simple Quantum Circuit involving more than one HADAMARD Gate

Let us analyze the following simple quantum circuit that involves the use of two HADAMARD gates.

Fig-6: A simple Quantum Circuit that involves the use of two HADAMARD Gates

On applying the first HADAMARD Gate to the computational basis quantum state $|0\rangle$, we receive

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

On applying the second HADAMARD Gate,

$$H \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \left(H|0\rangle + H|1\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} + \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{|0\rangle}{\sqrt{2}} + \cancel{\frac{|1\rangle}{\sqrt{2}}} + \frac{|0\rangle}{\sqrt{2}} - \cancel{\frac{|1\rangle}{\sqrt{2}}} \right)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{|0\rangle}{\sqrt{2}} + \frac{|0\rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{|0\rangle + |0\rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{2}{\sqrt{2}} |0\rangle \right) = \frac{2}{\sqrt{2}\cdot\sqrt{2}} |0\rangle$$

$$= \frac{2}{2} |0\rangle = |0\rangle$$

Figure:7 Effect of the Quantum Circuit, as depicted in Figure 6 above, on the computational basis quantum state $|0\rangle$

e-ISSN: 2582-5208
International Research Journal of Modernization in Engineering Technology and Science
Volume:02/Issue:05/May-2020 www.irjmets.com

Thus, the effect of the Quantum Circuit, as depicted in Figure 6 above, on the computational basis quantum state $|0\rangle$ results in the same output quantum state $|0\rangle$.

Let us now study the effect of the Quantum Circuit, as depicted in Figure 6 above, on the computational basis quantum state $|1\rangle$. On applying the first HADAMARD Gate to the computational basis quantum state $|1\rangle$, we receive

$$\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

On applying the second HADAMARD Gate,

$$H \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \left(H|0\rangle - H|1\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} - \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}} \left(\cancel{\frac{|0\rangle}{\sqrt{2}}} + \frac{|1\rangle}{\sqrt{2}} - \cancel{\frac{|0\rangle}{\sqrt{2}}} + \frac{|1\rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{|1\rangle}{\sqrt{2}} + \frac{|1\rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{|1\rangle + |1\rangle}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{2}} \left(\frac{2}{\sqrt{2}} |1\rangle \right)$$

$$= \frac{2}{\sqrt{2}\cdot\sqrt{2}} |1\rangle = \frac{\cancel{2}}{\cancel{2}} |1\rangle$$

$$= |1\rangle$$

Fig-8: Effect of the Quantum Circuit, as depicted in Figure 6 above, on the computational basis quantum state $|1\rangle$

Thus, the effect of the Quantum Circuit, as depicted in Figure 6 above, on the computational basis quantum state $|1\rangle$ results in the same output quantum state $|1\rangle$.

So, the effect of the Quantum Circuit, as depicted in Figure 6 above, on the computational basis quantum state $|0\rangle$ results in cancelling out the $|1\rangle$ states and reinforcing the $|0\rangle$ states. Similarly, the effect of the Quantum Circuit, as depicted in Figure 6 above, on the computational basis quantum state $|1\rangle$ results in cancelling out the $|0\rangle$ states and reinforcing the $|1\rangle$ states. Thus, the Quantum Circuit, as depicted in Figure 6 above, leaves both the $|0\rangle$ and $|1\rangle$ states unchanged. The net effect of the Quantum Circuit, as depicted in Figure 6 above, is, therefore, the same as that of a quantum wire.

e-ISSN: 2582-5208
International Research Journal of Modernization in Engineering Technology and Science
Volume:02/Issue:05/May-2020 www.irjmets.com

Fig-9: Net effect of the Quantum Circuit, as depicted in Figure 6 above, is the same as that of a Quantum Wire

Let us now assume that an arbitrary quantum state $|\psi\rangle$ is input to the Quantum Circuit, as depicted in Figure 6 above. On applying the first HADAMARD Gate to the arbitrary quantum state $|\psi\rangle$, we receive H $|\psi\rangle$. On applying the second HADAMARD Gate to H $|\psi\rangle$, we receive HH $|\psi\rangle$ as the output. The output HH $|\psi\rangle$ can be simply thought of as applying two H matrices to the arbitrary quantum state $|\psi\rangle$.

$$HH = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \cdot \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1\times1+1\times1 & 1\times1+1\times(-1)\\ 1\times1+(-1)\times1 & 1\times1+(-1)\times(-1)\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$$

$$= \begin{bmatrix}\cancel{2}\times\frac{1}{\cancel{2}} & 0\times\frac{1}{2}\\ 0\times\frac{1}{2} & \cancel{2}\times\frac{1}{\cancel{2}}\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Fig-10: Matrix Product HH

Therefore, computing the matrix product HH results in a 2 x 2 Identity Matrix. So, HH = I. We know that, upon multiplying a matrix A by the Identity Matrix I results in the same matrix A as the output. So, HH $|\psi\rangle$ = I $|\psi\rangle$ = $|\psi\rangle$. Thus, we have proved once again that the net effect of the Quantum Circuit, as depicted in Figure 6 above, is the same as that of a quantum wire.

e) A misconstrued assumption?

Suppose that we have set forth trying to discover a new Quantum Gate and have decided to experiment with, what we believe could be an interesting Quantum Gate (that we have decided to denote by J), quite like the HADAMARD Gate H. We have assumed that our new Quantum Gate J acts on the computational basis quantum states, $|0\rangle$ and $|1\rangle$ as follows.

$$J|0\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$$

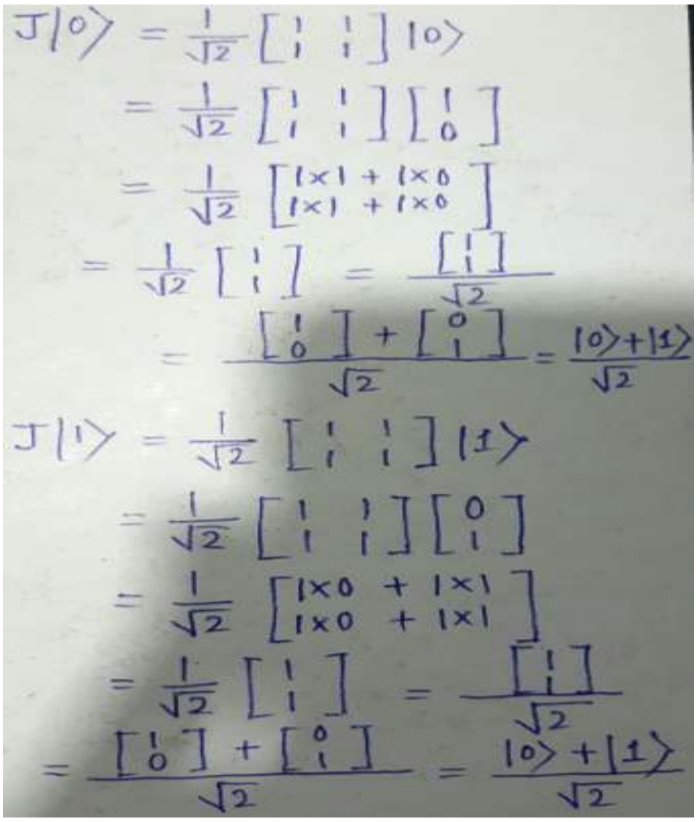
$$J|1\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$$

We have also assumed that the matrix representation of our new Quantum Gate J would be as follows.

$$J := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Fig-11: Matrix representation of our assumed new Quantum Gate J

Let us see if the matrix representation of our assumed new Quantum Gate J is correct.



$$J|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}|0\rangle$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1\times 1 + 1\times 0 \\ 1\times 1 + 1\times 0 \end{bmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\sqrt{2}}$$
$$= \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
$$J|1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}|1\rangle$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1\times 0 + 1\times 1 \\ 1\times 0 + 1\times 1 \end{bmatrix}$$
$$= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\sqrt{2}}$$
$$= \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\sqrt{2}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Fig-12: Mathematically proving that the matrix representation of our assumed new Quantum Gate J is correct

In fact, indeed, we have proved mathematically, as depicted in Figure 12 above, that the matrix representation of our assumed new Quantum Gate J is correct. So far so good – we are well on our way to discover a new Quantum Gate J. However, before celebrating our new discovery, let us do a final round of experiment – let us try to apply our assumed new Quantum Gate J to the following quantum state.

$$\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$J\,\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left(J|0\rangle - J|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} - \frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\cancel{\frac{|0\rangle}{\sqrt{2}}} + \cancel{\frac{|1\rangle}{\sqrt{2}}} - \cancel{\frac{|0\rangle}{\sqrt{2}}} - \cancel{\frac{|1\rangle}{\sqrt{2}}}\right)$$

$$= \frac{1}{\sqrt{2}} \cdot 0 = 0$$

Fig-13: Mathematically proving that our assumed new Quantum Gate J is not suitable for use as a Quantum Gate

So, we can see from the calculations in Figure 13 above that when we apply our assumed new Quantum Gate J to the following quantum state,

$$\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

both the computational basis quantum states $|0\rangle$ and $|1\rangle$ cancel each other out and we are left with no computational basis quantum states. This renders our assumed new Quantum Gate J unsuitable for use as a Quantum Gate.

III. CONCLUSION

This review paper provides the reader a detailed perspective of the HADAMARD gate H, which, we have seen helps expand the range of a qubit's quantum states and in doing so, the HADAMARD Gate helps in expanding the range of dynamical operations that a quantum computer can perform, which won't ever be possible on a conventional classical computer. We have seen how the HADAMARD Gate can be thought of as helping mix the computational basis quantum states $|0\rangle$ and $|1\rangle$ together.

ACKNOWLEDGEMENTS

The author sincerely bows down in all true respect, love and devotion before the ALMIGHTY GOD without whose grace and mercy, nothing is possible in this materialistic world.

The author wishes to express his true love for his better half, Mrs. Sukanya Basu Sarkar, who has always been a pillar of strength and inspiration to the author, and who probably is also one of the true and passionate critic and admirer of the author's work.

The author sincerely wishes his gratitude and love for the organization he is an extremely proud employee of, BMC Software India Pvt Ltd. I LOVE YOU, BMC!!!

And, finally, the author would like to express his humble and sincere gratitude to IBM Corporation, whose work on Quantum Computing and uncountable other technologies, has truly inspired the author to study and research on the field of Quantum Computing.

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